36710-36752, Fall 2020
Homework 1

Due Wed, Sept 16 by 5pm.

Questions 5, 6 and 7 are no longer part of this assignment. They will be on Homework 2

1. **Limits superior and inferior.**
   (a) Let A_n be $(-1/n, 1]$ if n is odd and $(-1, 1/n]$ if n is even. Find $\limsup_n A_n$ and $\liminf_n A_n$.
   (b) **Bonus Problem**. Let A_n the interior of the ball in $\mathbb{R}^2$ with unit radius and center $\left(\frac{(-1)^n}{n}, 0\right)$. Find $\limsup_n A_n$ and $\liminf_n A_n$.
   (c) Show that $(\limsup_n A_n)^c = \liminf_n A_n^c$ and $(\liminf_n A_n)^c = \limsup_n A_n^c$.

2. Let $\mathcal{A}$ be a collection of subsets of Ω. Let $\mathcal{F}$ be the intersection of all σ-fields that include $\mathcal{A}$ as a subset. Show that $\mathcal{F}$ is also a σ-field and it is the smallest σ-field that includes $\mathcal{A}$ as a subset.

3. Exercise 6 in Lecture Notes Set 1.
   Let $\mathcal{F}_1$, $\mathcal{F}_2, \ldots$ be classes of sets in a common space Ω such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for each n. Show that if each $\mathcal{F}_n$ is a field, then $\cup_{n=1}^{\infty} \mathcal{F}_n$ is also a field.

   If each $\mathcal{F}_n$ is a σ-field, then $\cup_{n=1}^{\infty} \mathcal{F}_n$ is not necessarily a σ-field. Think about the following case: Ω is the set of nonnegative integers and $\mathcal{F}_n$ is the σ-field consisting of the power set of $\{0, 1, \ldots, n\}$.
   Hint: You can prove this in more than one way. For instance show that the set of even numbers can be obtained as a countable unions of sets in $\bigcup_n \mathcal{F}_n$ but it cannot belong to $\bigcup_n \mathcal{F}_n$. Alternatively, show that the smallest σ-field containing $\bigcup_n \mathcal{F}_n$ must be 2^Ω, which is uncountable since Ω is infinite (...why?).

4. Let μ be a counting measure on a countably infinite set Ω. Show that there exists a decreasing sequence of sets $\{A_n\}$ such that $A_n \downarrow \emptyset$ but $\mu(A_n) = \infty$ for al n (and therefore $\lim_n \mu(A_n) \neq 0$). *(This should help addressing Exercise 13 in the lecture notes).*

5. **No longer part of this assignment!** Let $(\Omega, \mathcal{F})$ be a measurable space and $\{f_n\}_{n=1,2,\ldots}$ be a sequence of real valued measurable function on Ω. Show that the set $\{\omega \in \Omega \colon \lim_n f_n(\omega) \text{ exists}\}$ is measurable (i.e. it belongs to $\mathcal{F}$).

6. **No longer part of this assignment!** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $(S, \mathcal{A})$ a measurable space and $f \colon \Omega \to S$ a measurable function.. Show that, for arbitrary subsets $A, A_1, A_2, \ldots$ of S,

(a) $f^{-1}(A^c) = (f^{-1}(A))^c$;

(b) $f^{-1}(\cup_n A_n) = \bigcup_n f^{-1}(A_n)$ and

(c) $f^{-1}(\cap_n A_n) = \bigcap_n f^{-1}(A_n)$.

(The last two identities actually hold also for uncountable unions and intersections). Let $\mathcal{A}$ be a σ-field over S. Prove that the collection $f^{-1}(\mathcal{A}) = \{f^{-1}(A), A \in \mathcal{A}\}$ of subsets of Ω is a σ-field over Ω (in fact, the smallest σ-field on Ω that makes f measurable).

7. **No longer part of this assignment!** (**The induced measure is a measure**) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $(S, \mathcal{A})$ a measurable space and $f \colon \Omega \to S$ a measurable function. Show that the measure induced by f and μ, i.e. the function ν over $\mathcal{A}$ given by

$$A \mapsto \mu\left(f^{-1}(A)\right), \quad A \in \mathcal{A},$$

is a measure. Show by example that ν need not be σ-finite if μ is σ-finite.