36710-36752, Fall 2020
Homework 3

Due Oct 21, by 5pm.

1. Let μ be the counting measure on $(\mathbb{R}^1, \mathcal{B}^1)$. Let ν be the Lebesgue measure on $(\mathbb{R}^1, \mathcal{B}^1)$. Show that $\nu \ll \mu$ but there does not exist any function $f \geq 0$, such that $\nu(A) = \int_A f d\mu$ for all $A \in \mathcal{B}^1$. Why does Radon-Nikodym Theorem fail in this case?

2. Let μ_i be σ-finite measures, $i = 1, 2$. Show that if
$$\int \left[\int |f(\omega_1, \omega_2)| \, d\mu_1 \right] d\mu_2 < \infty$$
then
$$\int f d\mu_1 \times \mu_2 = \int \left[\int f(\omega_1, \omega_2) d\mu_1 \right] d\mu_2 = \int \left[\int f(\omega_1, \omega_2) d\mu_2 \right] d\mu_1.$$

3. Let X and Y be two independent, integrable random variables.
   (a) Show that $\mathbb{E}(XY) = \mathbb{E}X\mathbb{E}Y$.
   (b) Let f and g be two measurable functions (not necessarily distinct) from $\mathbb{R}$ to $\mathbb{R}$. Show that $f(X)$ and $g(Y)$ are independent. Therefore $\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)]$ if $f(X)$ and $g(Y)$ are integrable. *Hint: you may use, without proof, the fact that $\sigma(f(X)) \subset \sigma(X)$ and $\sigma(g(Y)) \subset \sigma(Y)$ (though you should think about why this is the case).*

4. Assume X and Y are integrable random variables. Prove that, for each $r > 0$,
$$\mathbb{E}|X + Y|^r \leq C_r \left(\mathbb{E}|X|^r + \mathbb{E}|Y|^r \right),$$
where $C_r = 1$ if $r \in (0, 1]$ and $C_r = 2^{r-1}$ for $r > 1$.
Hint: for $r > 1$ use Jensen's inequality. For $r \in (0, 1]$ use the fact that $(a+b)^r \leq a^r + b^r$ for all $a, b \geq 0$.

5. Prove Paley-Zygmund's inequality (which is a converse to Markov's inequality): let X be a non-negative random variable with finite variance. Then, for ay $\lambda > 0$,
$$\mathbb{P}(X \geq \lambda) \geq \frac{\left[(\mathbb{E}[X] - \lambda) + \right]^2}{\mathbb{E}[X^2]}.$$
If X is non-negative and bounded – that is, $0 \leq X \leq b$ almost surely for some $b > 0$ – prove that, for all $\lambda \in (0, \mathbb{E}[X])$,
$$\mathbb{P}(X \geq \lambda) \geq \frac{\mathbb{E}[X] - \lambda}{b - \lambda}.$$

6. Let $1 \leq p \leq q \leq \infty$. Assume that μ is a probability measure on $(\Omega, \mathcal{F})$ and let f be a measurable function. Show that $\|f\|_p \leq \|f\|_q$ (in particular, this implies that $\|f\|_\infty = \lim_{p \to \infty} \|f\|_p$.)