36710-36752, Fall 2020
Homework 4

Due Nov 11, by 5pm.

1. Consider the space $(\mathbb{R}^\infty, \mathcal{F}^\infty)$ of all infinite sequences of real numbers endowed with the Borel product σ-field. For any real number μ, let P_μ denote the distribution of an infinite sequence of i.i.d. random variables with the $N(\mu, 1)$ distribution. Show that, for any $\mu \neq \mu'$, P_μ and $P_{\mu'}$ are mutually singular: there exist sets A_μ and $A_{\mu'}$ in $\mathcal{F}^\infty$ such that $A_\mu \cap A_{\mu'} = \emptyset$ and $P_\mu(A_\mu^c) = P_{\mu'}(A_{\mu'}^c) = 0$. This is a feature of infinite i.i.d. Guassian sequences. In fact, when considering the distributions of finite i.i.d. Gaussian sequence, P_μ and $P_{\mu'}$ are always equivalent[1], for any $\mu \neq \mu'$ (no need to show this). *Hint: by the SLLN, the set* $A_\mu = \{(x_1, x_2, \ldots) \in \mathbb{R}^\infty \colon \lim_n \frac{1}{n}\sum_{i=1}^n x_i = \mu\}$ *has* P_μ*-probability* 1*...*

2. Show that $X_n \stackrel{a.s.}{\to} 0$ if and only if $\sup_{k \geq n} |X_k| \stackrel{p}{\to} 0$.

3. (WLLN under dependence.) Let $X_1, X_2, \ldots$ be a sequence of random variables with mean zero and such that $\mathbb{E}[X_n X_m] = \rho(|m - n|)$, where $\lim_{x\to\infty} \rho(x) = 0$ (notice that $\mathrm{Var}[X_n] = \rho(0)$ for all n). Show that

$$\frac{1}{n}\sum_{i=1}^n X_i \stackrel{P}{\to} 0.$$

4. Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of mean-zero, unit-variance random variables such that $\rho_n := \mathrm{Corr}(X_n, Y_n) \to 1$ as $n \to \infty$. Show that $X_n - Y_n \stackrel{P}{\to} 0$.

5. For a vector $x \in \mathbb{R}^d$, let $x(j)$ denote its jth coordinate. Let $\{X_n\}$ be a sequence of random vectors and X a random vector in $\mathbb{R}^d$. Show that $X_n \stackrel{P}{\to} X$ if and only if $X_n(j) \stackrel{P}{\to} X(j)$ for all $j = 1, \ldots, d$.

6. **In $\mathbb{R}^n$ most of the volume of the unit cube in $\mathbb{R}^n$ comes from the boundary of a ball of radius $\sqrt{n/3}$, when n is large.** Let $X = (X_1, X_2, \ldots, X_n)$ be vector in $\mathbb{R}^n$ comprised of independent random variables uniformly distributed on $[-1, 1]$. Then, for each $A \subset [-1, 1]^n$, $\Pr(X \in A)$ is the fraction of the volume of the unit cube $[-1, 1]^n$ occupied by A. (Notice that the volume of $[-1, 1]^n$ is 2^n.)

   (a) Show that, as $n \to \infty$,
$$\frac{\|X\|^2}{n} \stackrel{P}{\longrightarrow} \frac{1}{3}. \tag{1}$$
   (Recall that for $x = (x_1, \ldots, x) \in \mathbb{R}^n$, $\|x\|^2 = \sum_{i=1}^n x_i^2$).

[1] Recall that this means that P_μ and $P_{\mu'}$ are mutually absolutely continuous

(b) For any $\epsilon \in (0,1)$, consider the annulus

$$A_{\epsilon,n} = \left\{ x \in [-1,1]^n \colon (1-\epsilon)\sqrt{n/3} \le \|x\| \le \sqrt{n/3}(1+\epsilon) \right\}.$$

Use (1) to show that, for large n, almost all of the volume of $[-1,1]^n$ lies in $A_{\epsilon,n}$. This result should be surprising: when ϵ is minuscule and n is large, it says that most of the volume of $[-1,1]^n$ concentrates around a very thin annulus. This may seem wrong (draw the picture for the case of $n = 2$): how can a uniform distribution concentrate?!? In fact, this is a common, yet striking, features of probability distributions in high-dimensions.