36-710/752 Fall 2020

Advanced Probability Overview

Lecture 05 - Measurable Functions

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5.1 Measurable Functions

We restate the definition of a measurable function here for completeness

Definition 1 *Let $(\Omega, \mathcal{F})$ and $(S, \mathcal{A})$ be measurable spaces. Let $f : \Omega \to S$ be a function that satisfies $f^{-1}(A) \in \mathcal{F}$ for each $A \in \mathcal{A}$. Then we that f is $\mathcal{F}/\mathcal{A}$-measurable. If the σ-field's are to be understood from context, we simply say that f is measurable.*

Now, if we let $(\Omega, \mathcal{F}, P)$ be a probability space and $(S, \mathcal{A})$ be a measurable space, we might want to know if we can use $f : \Omega \to \mathbb{R}$ and P to construct a measure on $(S, \mathcal{A})$. Consider the following example:

Example 1 *Let $f : \Omega \to \mathbb{R}$ then, for some $A \in \mathcal{A}$, what is the probability that the image of f is in A?*

Although we will not go into the details here, we would also consider this setup when we try to define the notion of an integral. Returning to the definition of a measurable function in definition 1, we give two examples of measurable functions:

Example 2 *Let $\mathcal{F} = 2^{\Omega}$. Then every function from Ω to a set S is measurable no matter what $\mathcal{A}$ is.*

Example 3 *Let $\mathcal{A} = \{\varnothing, S\}$. Then every function from a set Ω to S is measurable, no matter what $\mathcal{F}$ is.*

5.2 Defining measurable functions by pre-images

Definition 1 can be restated in plainer English: if a function's pre-image of any measurable set is measurable then the function is measurable. But, why do we state this definition in terms of the pre-image? Why don't we use the more intuitive image? (Spoiler: because that doesn't work).

Remark 4 *A function f being measurable does not imply that for any $B \in \mathcal{F}$,*

$$f(B) = \{s \in S : s = f(\omega) \text{for some } \omega \in \Omega\}$$

is measurable.

We show this by example:

Example 5 *Consider Example 3 above. If $B \in \mathcal{F}$ such that $f(B) \neq \varnothing$ and $f(B) \neq S$ then $f(B)$ is not measurable.*

The inverse image is also useful because it commutes with union, complement, and intersection, which we outline in the following remark:

Remark 6 *If $\{A_\alpha\}_{\alpha \in I}$ is a collection of measurable subset of S, indexed by an arbitrary set I, then*

1. $f^{-1}(A^C) = [f^{-1}(A)]^C$,
2. $f^{-1}\left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup f^{-1}(A_\alpha)$, *and*
3. $f^{-1}\left(\bigcap_{\alpha \in I} A_\alpha\right) = \bigcap f^{-1}(A_\alpha)$.

See HW1 for a proof of these statements.

5.3 Measurable Functions and σ-fields

In this section, we build to a useful lemma that defines a measurable function in terms of the σ-field it generates. First, we define the σ-field generated by a measurable function:

Definition 2 *Let $f : \Omega \to S$, where $(S, \mathcal{A})$ is a measurable space. The σ-field $f^{-1}(\mathcal{A})$ is called the σ-field generated by f. The σ-field $f^{-1}(\mathcal{A})$ is also denoted $\sigma(f)$.*

Notice that $\sigma(f)$ is the smallest σ-field such that f is $\sigma(f)/\mathcal{A}$ measurable. Definition 2 leads us to the following important result:

Lemma 7 *Let $(\Omega, \mathcal{F})$ and $(S, \mathcal{A})$ be measurable spaces and $f : \Omega \to S$. Suppose that $\mathcal{A} = \sigma(\mathcal{C})$ for some collection of sets $\mathcal{C}$. Then f is $\mathcal{F}/\mathcal{A}$ measurable if and only if*

$$f^{-1}(\mathcal{C}) = \{f^{-1}(c), c \in \mathcal{C}\} \subseteq \mathcal{F}.$$

Proof: We will show the "if" direction. The collection on $\mathcal{A}' = \{A \in \mathcal{A} : f^{-1}(A) \in \mathcal{F}\}$ is a σ-field. By definition, $\mathcal{C} \subseteq \mathcal{A}'$. So, $\mathcal{A} = \sigma(\mathcal{C}) \subseteq \mathcal{A}'$. Also, by definition, $\mathcal{A}' \subseteq \mathcal{A}$. Therefore $\mathcal{A}' = \mathcal{A}$. ∎

Lemma 7 leads us to an important corollary that we can often use as shorthand for whether a function is measurable or not.

Corollary 8 *If f is a continuous function from one topological space to another (each with Borel σ-fields) then f is measurable.*

Proof: By continuity, $f^{-1}(\mathcal{U})$ is open whenever $\mathcal{U}$ is open. ∎

Remark 9 *In general, to check the measurability of functions taking values in $\mathbb{R}^k$, we only need to check that the pre-image of sets of the form $(-\infty, a]$ is measurable.*

Example 10 *Suppose $f : \mathbb{R} \to \mathbb{R}$, where the domain $\mathbb{R}$ is endowed by Borel σ-fields, then if f is monotone then f is measurable.*

Example 11 *Suppose that $f : \Omega \to \overline{\mathbb{R}}$ takes values in the extended reals. Then $f^{-1}(\{-\infty, \infty\}) = [f^{-1}((-\infty, \infty))]^C$. Also*

$$f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} \{\omega : f(\omega) > n\},$$

and similarly for $-\infty$. In order to check whether f is measurable we need to check that the inverse images for all semi-infinite intervals are measurable sets. If we include the infinite endpoint in these intervals, then we are done. If we do not include the infinite endpoint then we need to check that at least one of $\{\infty\}$ or $\{-\infty\}$ has measurable inverse image.

5.4 Multi-dimensional measurable functions

So far, we have only considered one-dimensional measurable functions. Intuitively, we might imagine there is relationship between whether a multi-dimensional function is measurable and whether its constituent one-dimensional functions are measurable. We state and prove this:

Theorem 12 *A function f of the form $f : (\Omega, \mathcal{F}) \to (\mathbb{R}^k, \mathcal{B}^k)$ is measurable if its k constituent functions are measurable. Stated differently,*

$$f(\omega) = \begin{bmatrix} f_1(\omega) \\ f_2(\omega) \\ \vdots \\ f_k(\omega) \end{bmatrix}$$

is measurable if and only if $f_i(\omega)$ is measurable for all $i = 1, \dots, k$.

Proof: Assume each f_i is measurable. Then, we are only concerned with the pre-image of hyper-rectangles because

$$\mathcal{B}^k = \sigma\big(\{(a_1, b_1] \times \cdots \times (a_k, b_k], a_i < b_i \ \forall \ i = 1, \dots, k\} \big).$$

Then, we have that

$$f^{-1}\big(\{(a_1, b_1] \times \cdots \times (a_k, b_k], a_i < b_i \ \forall \ i = 1, \dots, k\} \big) = \big\{\omega : f_i(\omega) \in (a_i, b_i] \ \forall \ i\big\} \tag{5.1}$$

$$= \bigcap_{i=1}^{k} f_i^{-1}\big((a_i, b_i]\big) \tag{5.2}$$

which is measurable because each $f_i^{-1}((a_i, b_i])$ is measurable.

Now, assume f is measurable. Then we claim that each set of the form $f^{-1}\big((a_i, b_i]\big)$ is measurable. To see this, consider equation 5.2. Then, fix a coordinate i. For all $j \neq i$, let $a_j \to -\infty$ and $b_j \to \infty$. Then

$$\bigcup_{l=1}^{k} f_l^{-1}\big((a_l, b_l]\big) \uparrow f_i^{-1}\big((a_i, b_i]\big).$$

So, $f_i^{-1}\big((a_i, b_i]\big)$ is measurable and f_i is measurable. ■

5.5 Properties of real-valued measurable functions

To conclude, we state three properties of real-valued measurable functions

Theorem 13 *Let $(\Omega, \mathcal{F}), (S, \mathcal{A})$, and $(T, \mathcal{B})$ be measurable spaces. Then*

1. *If f is measurable, then so is $a \cdot f$ for any $a \in \mathbb{R}$.*
2. *If $f : \Omega \to S$ and $g : S \to T$ are measurable, so is $g(f)$ or $g \circ f : \Omega \to T$*
3. *If f and g are real-valued functions, so are $f + g$, $\max\{f, g\}$, $\sqrt{|f - g|}$.*