36-710: Advanced Statistical Theory **Fall 2018**

Lecture 20: November 7

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

20.1 Spiked Covariance Model

References: Johnson & Lu (2009); Paul (2007); Nadler (2008).

$$\Sigma_{p\times p} = \nu\theta\theta^T + \mathbf{I}_d,$$

in which $\theta > 0, \nu \in \mathbb{S}^{p-1}$, ν leading eigenvector, $1+\theta$ leading eigenvalue.

Let $X_1,\ldots,X_n$ iid, $X_i \sim (0,\Sigma)$. $\hat{\Sigma} = \frac{1}{n}\sum_{i=1}^n x_i x_i^T, \hat{\nu}$ leading eigenvector of $\hat{\Sigma}$. $\sphericalangle(\nu,\hat{\nu}) \not\to 0$ with high probability unless $\frac{p}{n}\to 0$.

$$\min_{\epsilon\in\{-1,1\}} ||\epsilon\hat{\nu}-\nu|| \leq 2sin^2(\sphericalangle(\hat{\nu},\nu))$$

and by Davis-Kahan,

$$\min_{\epsilon\in\{-1,1\}} ||\epsilon\hat{\nu}-\nu|| \leq \frac{8}{\theta^2}||\hat{\Sigma}-\Sigma||_{op}$$

Assuming $X_i \in SG(||\Sigma||_{op})$,

$$\min_{\epsilon\in\{-1,1\}} ||\epsilon\hat{\nu}-\nu|| \lesssim \frac{1+\theta}{\theta} min\{\sqrt{\frac{p+log(1/\delta)}{n}}, \frac{p+log(1/\delta)}{n}\}.$$

20.2 Sparse PCA

$$\Sigma_{p\times p} = \nu\theta\theta^T + \mathbf{I}_d,$$

in which $\theta > 0$, $\nu \in \mathbb{S}^{p-1}$, $||\nu||_0 \leq k << n,p$, $\#\{i : \nu_i \neq 0\}$.

Estimate ν using $\hat{\nu}$ where $\hat{\nu}^T\hat{\Sigma}\hat{\nu} = \max_{u\in\mathbb{S}^{p-1},||u||_0\leq k'} u^T\hat{\Sigma}u, k \leq k' \leq \frac{p}{2} \to$ not computationally feasible!!!

Theorem 20.1 *Assume $X_1,\ldots,X_n \sim (0,\Sigma)$ such that $X_i \in SG_p(||\Sigma||op)$. Let $\hat{\nu}$ be a solution to the sparse PCA. Then*

$$\min_{\epsilon\in\{-1,1\}} ||\epsilon\hat{\nu}-\nu|| \leq \frac{1+\theta}{\theta}\max\sqrt{B_n}, B_n,$$

in which $B_n = [(k+k')log(ep/k+k') + log(1/\delta)]/n$, with probability $\geq 1-\delta$.

Proof: Let $s \subset \{1,\dots,p\}, A(S) = (A_{ij})_{i,j\in s}, x_s = (x_i)_{i\in s}$ and $s = supp(\hat{\nu}) \cup supp(\nu)$. We have that

$$\begin{aligned}
\theta sin^2(\sphericalangle(\hat{\nu},\nu)) &= \nu^T\Sigma\nu - \hat{\nu}^T\hat{\Sigma}\hat{\nu} \\
&= \nu^T\hat{\Sigma}\nu - \hat{\nu}^T\Sigma\hat{\nu} - \nu^T(\hat{\Sigma}-\Sigma)\nu \\
&\leq \hat{\nu}^T\hat{\Sigma}\nu - \hat{\nu}^T\Sigma\hat{\nu} - \nu^T(\hat{\Sigma}-\Sigma)\nu \\
&= \hat{\nu}^T(\hat{\Sigma}-\Sigma)\hat{\nu} - \nu^T(\hat{\Sigma}-\Sigma)\nu \\
&= <\hat{\Sigma}-\Sigma, \hat{\nu}\hat{\nu}^T - \nu\nu^T> \\
&= <\hat{\Sigma}(s)-\Sigma(s), \hat{\nu}_s\hat{\nu}_s^T - \nu_s\nu_s^T> \\
&\leq ||\hat{\Sigma}(s)-\Sigma(s)||_{op}||\hat{\nu}_s\hat{\nu}_s^T - \nu_s\nu_s^T||_1 \\
&\leq ||\hat{\Sigma}(s)-\Sigma(s)||_{op}\sqrt{2}||\hat{\nu}_s\hat{\nu}_s^T - \nu_s\nu_s^T||_2
\end{aligned}$$

Note that $rank(\hat{\nu}_s\hat{\nu}_s^T - \nu_s\nu_s^T) \leq 2$. Let's look separately at $\hat{\nu}_s\hat{\nu}_s^T - \nu_s\nu_s^T$.

$$\begin{aligned}
||\hat{\nu}_s\hat{\nu}_s^T - \nu_s\nu_s^T||_F &\leq ||\hat{\nu}\hat{\nu}^T - \nu\nu^T||_F \\
&= \sqrt{2sin^2(\sphericalangle(\hat{\nu},\nu))}.
\end{aligned}$$

Then, $sin(\sphericalangle(\hat{\nu},\nu)) \leq 2||\hat{\Sigma}(s)-\Sigma(s)||_{op}$, since $\min_{\epsilon\in\{-1,1\}}||\epsilon\hat{\nu}-\nu||^2 \leq sin(\sphericalangle(\hat{\nu},\nu))$.

We have shown that $\min_{\epsilon\in\{-1,1\}}||\epsilon\hat{\nu}-\nu||^2 \leq \frac{\sqrt{8}}{\theta}\sup_{s\in 1,\dots,p, s\neq\emptyset, |s|\leq k+k'}||\hat{\Sigma}(s)-\Sigma(s)||_{op}$.

$$\mathbb{P}(\sup_{s\in 1,\dots,p,s\neq\emptyset,|s|\leq k+k'}||\hat{\Sigma}(s)-\Sigma(s)||_{op} \geq t||\Sigma||_{op}) \lesssim \binom{p}{k+k'}q^{k+k'}exp-\frac{n}{2}\{(t/32)^2\wedge t/32\}$$

Recall the inequality $\binom{n}{k} \leq (\frac{en}{k})^k$.

Finish up the usual way.

20.3 Community detection in stochastic block model

Let $A_{n\times n}$ symmetric be the adjacency matrix, $A_{ij} = \mathbb{I}(i \text{ connected to } j), A_{ii} = 0$ all i.

20.3.1 Erdos - Renyi's Model

$$A_{ij} \sim Bernoulli(p), \text{ all } i < j.$$

More generally, one could assume $A_{ij} \sim Bernoulli(p_{ij})$ independent (inhomogeneous Bernoulli model).

20.3.2 Stochastic Block Model

Next class!