36-710: Advanced Statistical Theory **Fall 2018**

Lecture 4: September 12

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4.1 Last time

In previous lectures, we defined sub-Gaussian random variables by bounding their MGF:

$$X \in SG(\sigma^2) \implies \mathbb{E}\{\exp\{\lambda(X-\mu)\}\} \; \forall \lambda \in \mathbb{R}$$

where $\mu = \mathbb{E}\{X\}$. In particular, we derived the Hoeffding Inequality, which tells us that sub-Gaussian random variables have Gaussian tails. Other equivalent definitions of sub-Gaussianity can be found in Chapter 2 of Wainwright.

4.2 Today's lecture

Theorem 4.1 *Let $X \in SG(\sigma^2)$ with mean μ. It holds that*

$$\mathbb{E}\{|X|^p\} \le p(2\sigma^2)^{p/2}\Gamma\left(\frac{p}{2}\right) \implies \|X\|_p \lesssim \sigma\sqrt{p}$$

where last inequality is up to constant and holds for large p.

Proof:

$$\begin{aligned}\mathbb{E}\{|X|^p\} &= \int_0^\infty P(|X|^p \ge t)dt \\ &= \int_0^\infty P(|X| \ge t^{1/p})dt \\ &\le \int_0^\infty 2\exp\left\{-\frac{t^{2/p}}{2\sigma^2}\right\}dt \\ &= (2\sigma^2)^{p/2} p \underbrace{\int_0^\infty e^{-u}u^{p/2-1}du}_{\Gamma(\frac{p}{2})} \qquad \left[u = \frac{t^{2/p}}{2\sigma^2}\right]\end{aligned}$$

Note that $\Gamma\left(\frac{p}{2}\right) \le \left(\frac{p}{2}\right)^{\frac{p}{2}}$ and $p^{\frac{1}{p}} \le e^{\frac{1}{p}}$ for $p \ge 2$, hence:

$$\|X\|_p \le \sigma e^{\frac{1}{p}}\sqrt{p} \lesssim \sigma\sqrt{p}$$

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Special case:

$$X \sim N(0, \sigma^2) \implies \mathbb{E}\{|X|^p\} = \frac{\sigma^p 2^{p/2}\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}$$

Definition 4.2 (Sub-Exponential random variables) *A random variable is sub-Exponential (SE) with parameters ν^2 and α, where $\nu, \alpha > 0$, if the following holds:*

$$\mathbb{E}\{\exp\{\lambda(X-\mu)\}\} \leq \exp\left\{\frac{\lambda^2\nu^2}{2}\right\} \ \forall\lambda : |\lambda| < \frac{1}{\alpha}$$

Example 4.3 (χ_1^2 is $SE(4,4)$) *Let $Z \sim N(0,1)$ and consider $X = Z^2$, for which we have $\mathbb{E}\{X\} = 1$ and $X \sim \chi_1^2$. Then:*

$$\begin{aligned}
\mathbb{E}\{\exp\{\lambda(X-1)\}\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\{\lambda(z^2-1)\}\exp\left\{-\frac{z^2}{2}\right\} dz \\
&= \frac{e^{-\lambda}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left\{-\frac{z^2}{2}(1-2\lambda)\right\} dz \\
&= \frac{e^{-\lambda}}{\sqrt{2\pi}\sqrt{1-2\lambda}}\int_{-\infty}^{\infty} \exp\left\{-\frac{y^2}{2}\right\} dy \qquad \left[y = z\sqrt{1-2\lambda} \text{ for } \lambda < 1/2\right] \\
&= \frac{e^{-\lambda}}{\sqrt{1-2\lambda}}
\end{aligned}$$

We can use the following inequality:

$$-\log(1-u) - u \leq \frac{u^2}{2(u-1)} \quad u \in (0,1)$$

to conclude that

$$\mathbb{E}\{\exp\{\lambda(X-1)\}\} \leq \exp\left\{\frac{\lambda^2}{1-2\lambda}\right\} \quad |\lambda| < 1/2$$

This means that we can get the following one-tail bound:

$$P(Z^2 - 1 \geq 2t + 2\sqrt{t}) \leq e^{-t} \text{ for } t > 0$$

Notice that for $|\lambda| < 1/4$, we have:

$$\mathbb{E}\{\exp\{\lambda(X-1)\}\} \leq \exp\left\{\frac{\lambda^2}{1-2\lambda}\right\} \leq \exp\left\{\frac{4\lambda^2}{2}\right\}$$

so that $X \in SE(4,4)$.

4.2.1 Some properties of sub-Exponetial random variables

1. If $X \in SG(\sigma^2)$ and $\mathbb{E}\{X\} = 0$, then $X^2 - \mathbb{E}\{X^2\} \in SE(256\sigma^4, 16\sigma^2)$
2. If $\mathrm{Var}(X) \leq \sigma^2$ and $|X - \mu| \leq b$ a.e., then $X \in SE(2\sigma^2, 2b)$.

Proof:

$$\begin{aligned}
\mathbb{E}\left\{\exp\{\lambda(X-\mu)\}\right\} &\le 1 + \frac{\lambda^2\sigma^2}{2} + \sum_{k=3}^{\infty}\frac{\lambda^k}{k!}\mathbb{E}\{[X-\mu]^k\} && [\text{equality if } Var(X)=\sigma^2] \\
&\le 1 + \frac{\lambda^2\sigma^2}{2} + \frac{\lambda^2\sigma^2}{2}\sum_{k=3}^{\infty}[|\lambda|b]^{k-2} \\
&= 1 + \frac{\lambda^2\sigma^2}{2} + \frac{\lambda^2\sigma^2}{2}\sum_{k=0}^{\infty}[|\lambda|b]^{k} \\
&= 1 + \frac{\lambda^2\sigma^2}{2(1-|\lambda|b)} && [|\lambda| < 1/(2b)] \\
&\le \exp\left\{\frac{\lambda^2\sigma^2}{2(1-|\lambda|b)}\right\} && [1+x\le e^x] \\
&\le \exp\left\{\lambda^2\sigma^2\right\} && [|\lambda| < 1/(2b)]
\end{aligned}$$

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Remark: if $|X-\mu|\le b$ a.e., we have $X \in SG(\sigma^2)$.

Theorem 4.4 *(Sub-Exponential tail bound) Let $X \in SE(\nu^2,\alpha)$, then it holds that*

$$P(X-\mu \ge t) \le \begin{cases} \exp\left\{-\frac{t^2}{2\sigma^2}\right\} & \text{if } 0 < t \le \frac{\nu^2}{\alpha} \\ \exp\{-t\} & \text{if } t > \frac{\nu^2}{\alpha} \end{cases}$$

Proof: WLOG, take $\mathbb{E}\{X\}=0$. Fot $t>0$, we have

$$P(X\ge t) \le \exp\left\{-\lambda t + \frac{\lambda^2\nu^2}{2}\right\} \qquad [0<\lambda<1/\alpha]$$

The next step is to minimize over λ, that is to find $\inf_{\lambda\in[0,1/\alpha]}\exp\left\{-\lambda t+\frac{\lambda^2\nu^2}{2}\right\}$. *Without the constraint* on λ, the minimum occurs at $\lambda^* = t/\nu^2$, which is also the *constrained* minimum if $t<\nu^2/\alpha$. This yields a bound that is Gaussian-like. Otherwise, it is sufficient to note that $\exp\left\{-\lambda t+\frac{\lambda^2\nu^2}{2}\right\}$ is decreasing in $\lambda\in[0,\lambda^*]$, so that the constrained minimum occurs at $\lambda^* = t/\alpha$. Plugging λ^* into the log of the function we get

$$-\lambda^* t + \frac{\lambda^{*2}\nu^2}{2} = -\frac{t}{\alpha} + \frac{1}{2\alpha}\frac{\nu^2}{\alpha} \le -\frac{t}{2\alpha} \qquad \left[t \ge \frac{\nu^2}{\alpha}\right]$$

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Definition 4.5 (Bernstein Condition) *Let X be a random variable with $Var(X)\le\sigma^2$. X is said to satisfy the Bernstein Condition with parameter b if the following holds:*

$$\mathbb{E}\left\{|X-\mu|^k\right\} < \frac{1}{2}k!\sigma^2 b^{k-2} \quad \text{for } k\in\{3,4,\ldots\}$$

Result: if X satisfies the BC condition with parameter b, then it holds that

$$\mathbb{E}\left\{\exp\{\lambda(X-\mu)\}\right\} \le \exp\left\{\frac{\lambda^2\sigma^2}{2(1-|\lambda|b)}\right\} \quad \text{for } |\lambda| < \frac{1}{b}$$

which yields the following bound

$$P(|X - \mu| \geq t) \leq 2 \exp\left\{-\frac{t^2}{2(\sigma^2 + tb)}\right\}$$

In the proof of the bound, λ *is set to be* $\frac{t}{\sigma^2+bt} < \frac{1}{b}$.