# 36-710
# Homework 3

## Fall 2019

Due date: Wednesday 10/09 by 5pm in Alden's mailbox.

**Problem 1.** Before we state the problem let us familiarize ourselves with Gordon's inequality.

**Theorem 1** (Gordon's inequality). *Let $(X_{ut})_{u\in U,t\in T}$ and $(Y_{ut})_{u\in U,t\in T}$ be two mean-zero Gaussian processes indexed by pairs of points $(u,t)$ in a product set $U \times T$. Assume that we have*

$$\mathbb{E}(X_{ut} - X_{us})^2 \le \mathbb{E}(Y_{ut} - Y_{us})^2 \text{ for all } u, t, s,$$
$$\mathbb{E}(X_{ut} - X_{vs})^2 \ge \mathbb{E}(Y_{ut} - Y_{vs})^2 \text{ for all } u \neq v, \text{ and all } t, s.$$

*Then*

$$\mathbb{E} \inf_{u\in U} \sup_{t\in T} X_{ut} \le \mathbb{E} \inf_{u\in U} \sup_{t\in T} Y_{ut}.$$

Suppose now that $A$ is an $m \times n$ random matrix with independent $N(0,1)$ entries with $m \ge n$. Using Gordon's inequality prove that

$$\mathbb{E}s_n(A) \ge \sqrt{m} - \sqrt{n},$$

where $s_n(A)$ is the smallest singular value of $A$.
**Hint:** Recall the identity

$$s_n(A) = \min_{u\in S^{n-1}} \max_{v\in S^{m-1}} \langle Au, v\rangle.$$

Prove that

$$\mathbb{E}s_n(A) \ge \mathbb{E}\|h\|_2 - \mathbb{E}\|g\|_2, \text{ where } g \sim N(0, I_n), h \sim N(0, I_m).$$

Next, use the fact that $\mathbb{E}\|g\|_2 - \sqrt{n}$ is increasing in $n$. (You don't have to prove this, you may take it for granted).

**Problem 2.** Exercise 13.4 [W]

**Problem 3.** Exercise 13.5 [W]

**Problem 4.** Exercise 13.6 [W]

**Problem 5.** Exercise 13.8 [W].
**Hint:** You may assume that the metric entropy of the function class $\mathcal{F}$ satisfies $\log N(\delta, \mathcal{F}, \|\cdot\|_\infty) \asymp \frac{1}{\delta^2}$. This is because this class is similar to the class considered in Example 5.11 with $\alpha = 2$, $\gamma = 0$.

**Reading Exercise.** Read the paper "Sharp oracle inequalities for Least Squares estimators in shape restricted regression" by P. Bellec (https://arxiv.org/pdf/1510.08029.pdf). In particular pay attention to Theorem 2.3 and think about whether and how it is related to the "critical inequality".