**36-752, Spring 2018**
**Homework 1**

Due Thu Feb 15, by 5:00pm in Jisu's mailbox.

1. **Limits superior and inferior.**
   (a) Let $A_n$ be $(-1/n, 1]$ if $n$ is odd and $(-1, 1/n]$ if $n$ is even. Find $\limsup_n A_n$ and $\liminf_n A_n$.
   (b) **Bonus Problem**. Let $A_n$ the interior of the ball in $\mathbb{R}^2$ with unit radius and center $\left(\frac{(-1)^n}{n}, 0\right)$. Find $\limsup_n A_n$ and $\liminf_n A_n$.
   (c) Show that $\liminf_n A_n \subseteq \limsup_n A_n$
   (d) Show that $(\limsup_n A_n)^c = \liminf_n A_n^c$ and $(\liminf_n A_n)^c = \limsup_n A_n^c$.

2. Let $\mathcal{A}$ be a collection of subsets of $\Omega$. Let $\mathcal{F}$ be the intersection of all $\sigma$-fields that include $\mathcal{A}$ as a subset. Show that $\mathcal{F}$ is also a $\sigma$-field and it is the smallest $\sigma$-field that includes $\mathcal{A}$ as a subset.

3. Exercise 6 in Lecture Notes Set 1.
   Let $\mathcal{F}_1$, $\mathcal{F}_2, \ldots$ be classes of sets in a common space $\Omega$ such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for each $n$. Show that if each $\mathcal{F}_n$ is a field, then $\cup_{n=1}^{\infty} \mathcal{F}_n$ is also a field.

   If each $\mathcal{F}_n$ is a $\sigma$-field, then $\cup_{n=1}^{\infty} \mathcal{F}_n$ is not necessarily a $\sigma$-field. Think about the following case: $\Omega$ is the set of nonnegative integers and $\mathcal{F}_n$ is the $\sigma$-field of all subsets of $\{0, 1, \ldots, n\}$ and their complements.
   *Hint: You can prove this in more than one way. For instance show that the set of even numbers can be obtained as a countable unions of sets in $\bigcup_n \mathcal{F}_n$ but it cannot belong to $\bigcup_n \mathcal{F}_n$. Alternatively, show that the smallest $\sigma$-field containing $\bigcup_n \mathcal{F}_n$ must contain uncountably many sets but $\bigcup_n \mathcal{F}_n$ is countable.*

4. Let $\mu$ be a counting measure on an infinite set $\Omega$. Show that there exists a decreasing sequence of sets $A_n$ such that $A_n \downarrow \emptyset$ but $\lim_n \mu(A_n) \neq 0$. *(This should help addressing Exercise 13 in the lecture notes).*

5. If $\mu_1, \mu_2, \ldots$ are all measures on $(\Omega, \mathcal{F})$ and if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers, then $\sum_{n=1}^{\infty} a_n \mu_n$ is a measure on $(\Omega, \mathcal{F})$.

6. Let $A_1, \ldots, A_n$ be arbitrary subsets of $\Omega$. Describe as explicitly as you can $\mathcal{F}_n$, the smallest $\sigma$-field containing them. Give a non-trivial upper bound on the cardinality of $\mathcal{F}_n$. List all the elements of $\mathcal{F}_2$.

7. Let $\mu$ be a finite measure on $(\mathbb{R}, \mathcal{B})$ and, for any $x \in \mathbb{R}$, set $F(x) = \mu\left((-\infty, x]\right)$. Show that $F$ is cádlág.

8. Let $f \colon \Omega \to S$. Show that, for arbitrary subsets $A, A_1, A_2, \ldots$ of $S$,

(a) $f^{-1}(A^c) = (f^{-1}(A))^c$;

(b) $f^{-1}(\cup_n A_n) = \bigcup_n f^{-1}(A_n)$ and

(c) $f^{-1}(\cap_n A_n) = \bigcap_n f^{-1}(A_n)$.

(The last two identities actually hold also for uncountable unions and intersections). Let $\mathcal{A}$ be a $\sigma$-field over $S$. Prove that the collection $f^{-1}(\mathcal{A}) = \{f^{-1}(A), A \in \mathcal{A}\}$ of subsets of $\Omega$ is a $\sigma$-field over $\Omega$ (in fact, the smallest $\sigma$-field on $\Omega$ that makes $f$ measurable).

9. Let $(\Omega, \mathcal{F})$ be a measurable space and $\{f_n\}_{n=1,2,\ldots}$ be a sequence of real valued measurable function on $\Omega$. Show that the set $\{\omega \in \Omega \colon \lim_n f_n(\omega) \text{ exists}\}$ is measurable (i.e. it belongs to $\mathcal{F}$).

10. (**The induced measure is a measure**) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $(S, \mathcal{A})$ a measurable space and $f \colon \Omega \to S$ a measurable function. Show that the measure induced by $f$ and $\mu$, i.e. the function $\nu$ over $\mathcal{A}$ given by

$$A \mapsto \mu\left(f^{-1}(A)\right), \quad A \in \mathcal{A},$$

is a measure. Show by example that $\nu$ need not be $\sigma$-finite if $\mu$ is $\sigma$-finite.