36-752, Spring 2018
Homework 1 Solution

Due Thu Feb 15, by 5:00pm in Jisu's mailbox.

Points: 100+5 pts total for the assignment.

1. **Limits superior and inferior.**

   (a) Let A_n be $(-1/n, 1]$ if n is odd and $(-1, 1/n]$ if n is even. Find $\limsup_n A_n$ and $\liminf_n A_n$.

   (b) **Bonus Problem**. Let A_n the interior of the ball in $\mathbb{R}^2$ with unit radius and center $\left(\frac{(-1)^n}{n}, 0\right)$. Find $\limsup_n A_n$ and $\liminf_n A_n$.

   (c) Show that $\liminf_n A_n \subseteq \limsup_n A_n$

   (d) Show that $(\limsup_n A_n)^c = \liminf_n A_n^c$ and $(\liminf_n A_n)^c = \limsup_n A_n^c$.

Points: 10 + 5 pts = 4 + 5 + 3 + 3.

Solution.

(a)

Note that for any $k \in \mathbb{N}$, $A_k \cup A_{k+1} = (-1, 1]$. Hence $\bigcup_{k=n}^{\infty} A_k = (-1, 1]$ for all $n \in \mathbb{N}$, and hence

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} (-1, 1] = (-1, 1].$$

Also, note that for any $m \in \mathbb{N}$ $\bigcap_{k=m}^{\infty} A_{2k-1} = [0, 1]$ and $\bigcap_{k=m}^{\infty} A_{2k} = (-1, 0]$. Hence $\bigcap_{k=n}^{\infty} A_k = \{0\}$ for any $n \in \mathbb{N}$, and hence

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \{0\} = \{0\}.$$

(b)

Let $D := \{x \in \mathbb{R}^2 : \|x\|_2 < 1\}$ and $B := \{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\|_2 = 1, x_1 \neq 0\}$. We will show that $\liminf_n A_n = D$ and $\limsup_n A_n = D \cup B$.

For $\liminf_n A_n$, note that $x \in \liminf_n A_n$ if and only if $x \in A_n$ for all but finite n. Suppose $x \in D$. Then $\|x\|_2 < 1$, so choose N large enough so that $\frac{1}{N} < 1 - \|x\|_2$. Then for all $n \geq N$,

$$\begin{aligned}\left\| x - \left(\frac{(-1)^n}{n}, 0\right) \right\|_2 &\leq \|x\|_2 + \left\| \left(\frac{(-1)^n}{n}, 0\right) \right\|_2 \\ &= \|x\|_2 + \frac{1}{n} \leq \|x\|_2 + \frac{1}{N} < 1.\end{aligned}$$

Then $x \in A_n$ for all $n \geq N$, and hence $x \in \liminf_n A_n$, which implies $D \subset \liminf_n A_n$. Now, suppose $x \notin D$ and $x_1 \geq 0$. Then for all odd n,

$$\left\| x - \left(\frac{(-1)^n}{n},\, 0 \right) \right\|_2 = \left\| \left(x_1 - \frac{1}{n},\, x_2 \right) \right\|_2 > \|(x_1,\, x_2)\|_2 \geq 1,$$

Hence $x \notin A_n$ for all odd n, and hence $x \notin \liminf_n A_n$. Similarly, when $x \notin D$ and $x_1 \leq 0$, then $x \notin A_n$ for all even n, and hence $x \notin \liminf_n A_n$. These imply $\liminf_n A_n \subset D$, and hence

$$\liminf_n A_n = D.$$

For $\limsup_n A_n$, note that $x \in \limsup_n A_n$ if and only if $x \in A_n$ for infinitely many n. Suppose $x \in D \cup B$. We have already shown that $D = \liminf_n A_n \subset \limsup_n A_n$, and hence if $x \in D$ then $x \in \limsup_n A_n$. Now, suppose $x \in B$ and $x_1 > 0$. Then $\|x\|_1 = 1$. Choose N large enough so that $\frac{1}{N} < |x_1|$. Then for all even n with $n \geq N$, $\left|x_1 - \frac{1}{n}\right| \leq |x_1|$, and hence

$$\begin{aligned} \left\| x - \left(\frac{(-1)^n}{n}, 0 \right) \right\|_2 &= \left\| \left(x_1 - \frac{1}{n},\, x_2 \right) \right\|_2 \\ &< \|(x_1,\, x_2)\|_2 = 1. \end{aligned}$$

Hence $x \in A_n$ for all even n with $n \geq N$, and hence $x \in \limsup_n A_n$. Similarly, when $x \in B$ and $x_1 < 0$, $x \in A_n$ for all odd n with $n \geq N$, and hence $x \in \limsup_n A_n$. These imply that $D \cup B \subset \limsup_n A_n$. Now, suppose $x \notin D \cup B$. Then $\|x\|_2 > 1$ or $x = (0, \pm 1)$. When $\|x\|_2 > 1$, choose N large enough so that $\frac{1}{N} < 1 - \|x\|_2$. Then for all $n \geq N$,

$$\begin{aligned} \left\| x - \left(\frac{(-1)^n}{n}, 0 \right) \right\|_2 &\geq \|x\|_2 - \left\| \left(\frac{(-1)^n}{n}, 0 \right) \right\|_2 \\ &= \|x\|_2 - \frac{1}{n} \geq \|x\|_2 - \frac{1}{N} > 1. \end{aligned}$$

Then $x \notin A_n$ for all n with $n \geq N$, and hence $x \notin \limsup_n A_n$. Also, when $x = (0, \pm 1)$, then for all n,

$$\left\| x - \left(\frac{(-1)^n}{n}, 0 \right) \right\|_2 = \left\| \left(-\frac{(-1)^n}{n}, \pm 1 \right) \right\|_2 = \sqrt{1 + \frac{1}{n^2}} > 1,$$

Then $x \notin A_n$ for all n, and hence $x \notin \limsup_n A_n$. These show $\limsup_n A_n \subset D \cup B$, and hence

$$\limsup_n A_n = D \cup B.$$

(c)

Note that for all $m, n \in \mathbb{N}$,

$$\bigcap_{k=m}^{\infty} A_k \subset A_{\max\{m,n\}} \subset \bigcup_{k=n}^{\infty} A_k.$$

Hence for all $n \in \mathbb{N}$,

$$\liminf_n A_n = \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} A_k \subset \bigcup_{k=n}^{\infty} A_k$$

holds, and hence

$$\liminf_n A_n \subset \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \limsup_n A_n.$$

(d)
By applying De Morgan's law,

$$\left(\limsup_n A_n\right)^{\complement} = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right)^{\complement} = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k\right)^{\complement} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^{\complement} = \liminf_n A_n^{\complement}.$$

Similarly,

$$\left(\liminf_n A_n\right)^{\complement} = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right)^{\complement} = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k\right)^{\complement} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^{\complement} = \liminf_n A_n^{\complement}.$$

2. Let $\mathcal{A}$ be a collection of subsets of Ω. Let $\mathcal{F}$ be the intersection of all σ-fields that include $\mathcal{A}$ as a subset. Show that $\mathcal{F}$ is also a σ-field and it is the smallest σ-field that includes $\mathcal{A}$ as a subset.

Points: 10 pts.

Solution.
Let $\mathscr{A} := \{\mathcal{G} \subset 2^{\Omega} : \mathcal{G} \text{ is } \sigma\text{-field}, \mathcal{A} \subset \mathcal{G}\}$, and let $\mathcal{F} = \bigcap_{\mathcal{G} \in \mathscr{A}} \mathcal{G}$.
We will first show that $\mathcal{F}$ is a σ-field. First, $\Omega \in \mathcal{G}$ for any σ-field $\mathcal{G}$, and hence $\Omega \in \mathcal{F}$ as well. Second, if $A \in \mathcal{F}$, then $A \in \mathcal{G}$ for all $\mathcal{G} \in \mathscr{A}$. Then from $\mathcal{G}$ being a σ-field, $A^{\complement} \in \mathcal{G}$ for all $\mathcal{G} \in \mathscr{A}$, and hence $A^{\complement} \in \mathcal{F}$. Third, suppose $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$, then $\{A_n\}_{n=1}^{\infty} \subset \mathcal{G}$ for all $\mathcal{G} \in \mathscr{A}$. Then from $\mathcal{G}$ being a σ-field, $\bigcup_n A_n \in \mathcal{G}$ for all $\mathcal{G} \in \mathscr{A}$, and hence $\bigcup_n A_n \in \mathcal{F}$ as well. Hence $\mathcal{F}$ is a σ-field.
Now, note that $\mathcal{A} \subset \mathcal{G}$ for all $\mathcal{G} \in \mathscr{A}$, and hence $\mathcal{A} \subset \mathcal{F}$, and $\mathcal{F}$ is a σ-field with $\mathcal{A} \subset \mathcal{F}$. Also, for any $\mathcal{G}$: σ-field with $\mathcal{A} \subset \mathcal{G}$. Then $\mathcal{G} \in \mathscr{A}$, and hence $\mathcal{F} \subset \mathcal{G}$. Hence $\mathcal{F}$ is the smallest σ-field that includes $\mathcal{A}$ as a subset.

3. Exercise 6 in Lecture Notes Set 1.
Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be classes of sets in a common space Ω such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for each n. Show that if each $\mathcal{F}_n$ is a field, then $\cup_{n=1}^{\infty}\mathcal{F}_n$ is also a field.

   If each $\mathcal{F}_n$ is a σ-field, then $\cup_{n=1}^{\infty}\mathcal{F}_n$ is not necessarily a σ-field. Think about the following case: Ω is the set of nonnegative integers and $\mathcal{F}_n$ is the σ-field of all subsets of $\{0, 1, \ldots, n\}$ and their complements.
Hint: You can prove this in more than one way. For instance show that the set of even numbers can be obtained as a countable unions of sets in $\bigcup_n \mathcal{F}_n$ *but it cannot belong to* $\bigcup_n \mathcal{F}_n$. *Alternatively, show that the smallest* σ*-field containing* $\bigcup_n \mathcal{F}_n$ *must contain uncountably many sets but* $\bigcup_n \mathcal{F}_n$ *is countable.*

   Points: 10 pts.

   Solution.
We will first show that $\bigcup_n \mathcal{F}_n$ is a field. First, $\Omega \in \mathcal{F}_n$ for any n, and hence $\Omega \in \bigcup_n \mathcal{F}_n$. Second, if $A \in \bigcup_n \mathcal{F}_n$, then there exists $n \in \mathbb{N}$ with $A \in \mathcal{F}_n$. Then from $\mathcal{F}_n$ being a field, $A^{\complement} \in \mathcal{F}_n$, and hence $A^{\complement} \in \bigcup_n \mathcal{F}_n$ as well. Third, suppose $A, B \in \bigcup_n \mathcal{F}_n$. Then there exists $m, n \in \mathbb{N}$ with $A \in \mathcal{F}_m$ and $B \in \mathcal{F}_n$. Then from $\mathcal{F}_m, \mathcal{F}_n \subset \mathcal{F}_{\max\{m,n\}}$, $A, B \in \mathcal{F}_{\max\{m,n\}}$ as well. Then from $\mathcal{F}_{\max\{m,n\}}$ being a field, $A \cup B \in \mathcal{F}_{\max\{m,n\}}$, and hence $A \cup B \in \bigcup_n \mathcal{F}_n$ as well. Hence $\bigcup_n \mathcal{F}_n$ is a field.

   For counterexample for σ-field, let $[n] := \{1, \ldots, n\}$, and let $\mathcal{F}_n := \{A \subset \mathbb{N} :$ either $A \subset [n]$ or $\mathbb{N}\backslash A \subset [n]\}$. Then $\mathcal{F}_n$ is σ-field. First, $\mathbb{N}\backslash\mathbb{N} = \emptyset \subset [n]$, and hence $\mathbb{N} \in \mathcal{F}_n$. Second, if $A \in \mathcal{F}_n$, then either $A \subset [n]$ or $\mathbb{N}\backslash A \subset [n]$, and then $A^{\complement}$ satisfies either $\mathbb{N}\backslash A^{\complement} = A \subset [n]$ or $A^{\complement} = \mathbb{N}\backslash A \subset [n]$, and hence $A^{\complement} \in \mathcal{F}_n$. Third, if $\{A_m\}_{m=1}^{\infty} \subset \mathcal{F}_n$, then there are two cases: if $A_m \subset [n]$ for all m, then $\bigcup_m A_m \subset [n]$ as well, and hence $\bigcup_m A_m \in \mathcal{F}_n$. If there exists m such that $\mathbb{N}\backslash A_m \subset [n]$, then $\mathbb{N}\backslash(\bigcup_m A_m) \subset \mathbb{N}\backslash A_m \subset [n]$, and hence $\bigcup_m A_m \in \mathcal{F}_n$. Hence, $\mathcal{F}_n$ is a σ-field. Also, $A \subset [n]$ or $\mathbb{N}\backslash A \subset [n]$ implies $A \subset [n+1]$ or $\mathbb{N}\backslash A \subset [n+1]$, and hence $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.

   Now, consider $A_m = \{2m\}$ for $m \in \mathbb{N}$, so that $\bigcup_m A_m = \{2, 4, \ldots\}$. Then $A_m \in \mathcal{F}_{2m} \subset \bigcup_m \mathcal{F}_m$ for all $m \in \mathbb{N}$. However, if we assume that $\bigcup_m A_m \in \bigcup_n \mathcal{F}_n$, then there exists a finite n such that $\bigcup_m A_m \in \mathcal{F}_n$. This implies that either $\bigcup_m A_m = \{2, 4, \ldots\} \subset [n]$ or $\mathbb{N}\backslash(\bigcup_m A_m) = \{1, 3, \ldots\} \subset [n]$. But this is impossible since both $\bigcup_m A_m$ and $\mathbb{N}\backslash(\bigcup_m A_m)$ are infinite sets. Hence $\bigcup_m A_m \notin \bigcup_n \mathcal{F}_n$, and hence $\bigcup_n \mathcal{F}_n$ is not a σ-field.

4. Let μ be a counting measure on an infinite set Ω. Show that there exists a decreasing sequence of sets A_n such that $A_n \downarrow \emptyset$ but $\lim_n \mu(A_n) \neq 0$. *(This should help addressing Exercise 13 in the lecture notes).*

   Points: 10 pts.

   Solution.

Since Ω is infinite, there exists an injective function $f : \mathbb{N} \to \Omega$. Let $A_n := \{f(m) \in \Omega : m \in \mathbb{N}, m \geq n\}$. Then $A_n \supset A_{n+1}$. Also, $A_n \subset f(\mathbb{N}) := \{f(m) \in \Omega : m \in \mathbb{N}\}$, and for any $f(m) \in f(\mathbb{N})$, $f(m) \notin A_n$ for $n > m$, and hence $A_n \downarrow \emptyset$. However, all A_n's are infinite sets, so $\mu(A_n) = \infty$ for all n, and hence $\lim_{n\to\infty} \mu(A_n) = \infty \neq 0$.

5. If $\mu_1, \mu_2, \ldots$ are all measures on $(\Omega, \mathcal{F})$ and if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers, then $\sum_{n=1}^{\infty} a_n\mu_n$ is a measure on $(\Omega, \mathcal{F})$.

Points: 10 pts.

Solution.

We check whether $\sum_{n=1}^{\infty} a_n\mu_n$ is a measure. First, for any $A \in \mathcal{F}$, $\mu_n(A) \geq 0$ from μ_n being a measure and $a_n \geq 0$ for all n, and hence

$$\left(\sum_{n=1}^{\infty} a_n\mu_n\right)(A) = \sum_{n=1}^{\infty} a_n\mu_n(A) \geq 0.$$

Second, for all $n \in \mathbb{N}$, $\mu_n(\emptyset) = 0$ from μ_n being a measure, and hence

$$\left(\sum_{n=1}^{\infty} a_n\mu_n\right)(\emptyset) = \sum_{n=1}^{\infty} a_n\mu_n(\emptyset) = 0.$$

Third, let $A_1, A_2, \ldots$ be disjoint sets in $\mathcal{F}$. Then from countable additivity of each μ_n and Fubini's theorem,

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} a_n\mu_n\right)\left(\bigcup_{m=1}^{\infty} A_m\right) &= \sum_{n=1}^{\infty} a_n\mu_n\left(\bigcup_{m=1}^{\infty} A_m\right) \\
&= \sum_{n=1}^{\infty} a_n \sum_{m=1}^{\infty} \mu_n(A_m) \text{ (countable additivity)} \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_n\mu_n(A_m) \text{ (Fubini's theorem)} \\
&= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_n\mu_n\right)(A_m).
\end{aligned}$$

Hence countable additivity holds for $\sum_{n=1}^{\infty} a_n\mu_n$ as well. And hence $\sum_{n=1}^{\infty} a_n\mu_n$ is a measure.

6. Let $A_1, \ldots, A_n$ be arbitrary subsets of Ω. Describe as explicitly as you can $\mathcal{F}_n$, the smallest σ-field containing them. Give a non-trivial upper bound on the cardinality of $\mathcal{F}_n$. List all the elements of $\mathcal{F}_2$.

Points: 10 pts.

Solution.

For any $\omega \in \{-1,1\}^n$ so that w_i is either -1 or 1, let $A_\omega := \left(\bigcap_{i:\omega_i=1} A_i\right) \cap \left(\bigcap_{i:\omega_i=-1} A_i^{\complement}\right)$. And for any $I \subset 2^{\{-1,1\}^n}$, let $A_I = \bigcup_{\omega \in I} A_\omega$. And let

$$\begin{aligned}\mathcal{G}_n &:= \left\{A_I : I \subset 2^{\{-1,1\}^n}\right\} \\ &= \left\{A_{\omega^{(1)}} \cup \cdots \cup A_{\omega^{(k)}} : \omega^{(j)} \in \{-1,1\}^n\right\}.\end{aligned}$$

Note that if we let $I_i := \{\omega \in \{-1,1\}^n : \omega_i = 1\}$, then $A_i = A_{I_i}$ and $A_i^{\complement} = A_{2^{\{-1,1\}^n} \setminus I_i}$. Also, all A_ω are disjoint.

We first show that $\mathcal{G}_n$ is a σ-field. First, fix any $i \in I$, then from $A_i = A_{I_i}$ and $A_i^{\complement} = A_{2^{\{-1,1\}^n} \setminus I_i}$,

$$\Omega = A_i \cup A_i^{\complement} = A_{I_i} \cup A_{2^{\{-1,1\}^n} \setminus I_i} = A_{2^{\{-1,1\}^n}} \in \mathcal{G}_n.$$

Second, for any $\omega \in \{-1,1\}^n$, let $I_\omega := 2^{\{-1,1\}^n} \setminus \{\omega\}$, then $A_\omega^{\complement} = A_{I_\omega}$, and hence if $A_I \in \mathcal{G}_n$, then from A_ω's being disjoint,

$$\begin{aligned}A_I^{\complement} &= \left(\bigcup_{\omega \in I} A_\omega\right)^{\complement} = \bigcap_{\omega \in I} A_\omega^{\complement} \\ &= \bigcap_{\omega \in I} A_{I_\omega} = A_{\bigcap_{\omega \in I} I_\omega} \in \mathcal{G}_n.\end{aligned}$$

Third, if $\{A_{J_i}\}_{i=1}^{\infty} \subset \mathcal{G}_n$, then

$$\bigcup_{i=1}^{\infty} A_{J_i} = A_{\bigcup_{i=1}^{\infty} J_i} \in \mathcal{G}_n.$$

Hence $\mathcal{G}_n$ is a σ-field. Also, suppose $\mathcal{F}$ is a σ-field containing $A_1, \ldots, A_n$. Then since σ-field is closed under finite intersection and union, $\mathcal{G}_n \subset \mathcal{F}$. Hence $\mathcal{G}_n$ is the smallest σ-field containing $A_1, \ldots, A_n$, i.e.

$$\mathcal{F}_n = \mathcal{G}_n = \left\{A_I : I \subset 2^{\{-1,1\}^n}\right\}.$$

Then, since $|2^{\{-1,1\}^n}| = 2^{2^n}$,

$$|\mathcal{F}_n| \leq 2^{2^n}.$$

This bound is actually strict: suppose $\Omega = \{-1,1\}^n$ and $A_i = \{\omega \in \{-1,1\}^n : \omega_i = 1\}$. Then for any $\omega \in \{-1,1\}^n$, $A_\omega = \left(\bigcap_{i:\omega_i=1} A_i\right) \cap \left(\bigcap_{i:\omega_i=-1} A_i^{\complement}\right) = \{\omega\}$, hence A_ω are all disjoint singleton sets. And hence for each $I \subset 2^{\{-1,1\}^n}$, corresponding $A_I = \bigcup_{\omega \in I} A_\omega$'s are all different as well, and hence $|\mathcal{F}_n| = 2^{2^n}$ for this case.

From the above construction, $\mathcal{F}_2$ consists of arbitrary numbers of unions from $A_1 \cap A_2, A_1 \cap A_2^\complement, A_1^\complement \cap A_2, A_1^\complement \cap A_2^\complement$. The numbers range from 0 to 4. Taking 0 union only gives $\emptyset$, and 1 union gives themselves. Taking 2 unions gives A_1, A_2, $A_1^\complement$, $A_2^\complement$, $(A_1 \cap A_2^\complement) \cup (A_1^\complement \cap A_2)$, $(A_1 \cap A_2) \cup (A_1^\complement \cap A_2^\complement)$. Taking 3 union gives $A_1 \cup A_2$, $A_1 \cup A_2^\complement$, $A_1^\complement \cup A_2$, $A_1^\complement \cup A_2^\complement$. And taking 4 union gives Ω. Hence $\mathcal{F}_2$ can be enlisted as

$$\begin{aligned}\mathcal{F}_2 = \{&\emptyset, A_1 \cap A_2, A_1 \cap A_2^\complement, A_1^\complement \cap A_2, A_1^\complement \cap A_2^\complement,\\ &A_1, A_2, A_1^\complement, A_2^\complement, (A_1 \cap A_2^\complement) \cup (A_1^\complement \cap A_2), (A_1 \cap A_2) \cup (A_1^\complement \cap A_2^\complement),\\ &A_1 \cup A_2, A_1 \cup A_2^\complement, A_1^\complement \cup A_2, A_1^\complement \cup A_2^\complement, \Omega\}.\end{aligned}$$

7. Let μ be a finite measure on $(\mathbb{R}, \mathcal{B})$ and, for any $x \in \mathbb{R}$, set $F(x) = \mu((-\infty, x])$. Show that F is cádlág.

Points: 10 pts.

Solution.

We need to check that for all $x \in \mathbb{R}$, the left limit $\lim_{y\uparrow x} F(y)$ exists, and the right limit $\lim_{y\downarrow x} F(y)$ exists and equals $F(x)$. For the left limit, we first show $\lim_{n\to\infty} F\left(x - \frac{1}{n}\right) = \mu((-\infty, x))$. Note that $\left\{\left(-\infty, x - \frac{1}{n}\right]\right\}_{n=1}^\infty$ is a monotonically increasing sequence and $\lim_{n\to\infty}\left(-\infty, x - \frac{1}{n}\right] = (-\infty, x)$, and hence from monotonicity of the measure,

$$\begin{aligned}\lim_{n\to\infty} F\left(x - \frac{1}{n}\right) &= \lim_{n\to\infty} \mu\left(\left(-\infty, x - \frac{1}{n}\right]\right)\\ &= \mu\left(\lim_{n\to\infty}\left(-\infty, x - \frac{1}{n}\right]\right)\\ &= \mu\left((-\infty, x)\right).\end{aligned}$$

Now, from F being non-decreasing function, $\liminf_{y\uparrow x} F(y) \geq F\left(x - \frac{1}{n}\right)$ for any $n \in \mathbb{N}$, and hence $\liminf_{y\uparrow x} F(y) \geq \lim_{n\to\infty} F\left(x - \frac{1}{n}\right) = \mu\left((-\infty, x)\right)$. Also for all $y < x$, $F(y) = \mu((-\infty, y]) \leq \mu((-\infty, x))$, and hence $\limsup_{y\uparrow x} F(y) \leq \mu((-\infty, x))$. And hence

$$\lim_{y\uparrow x} F(y) = \liminf_{y\uparrow x} F(y) = \limsup_{y\uparrow x} F(y) = \mu\left((-\infty, x)\right),$$

and hence the left limit $\lim_{y\uparrow x} F(y)$ exists.

Similarly for the right limit, we first show $\lim_{n\to\infty} F\left(x + \frac{1}{n}\right) = F(x)$. Note that $\left\{\left(-\infty, x + \frac{1}{n}\right]\right\}_{n=1}^\infty$ is a monotonically increasing sequence and $\lim_{n\to\infty}\left(-\infty, x + \frac{1}{n}\right] = (-\infty, x]$, and note that $\mu\left((-\infty, x + 1]\right) < \infty$ from the finiteness of μ. Hence from monotonicity of the measure,

$$\begin{aligned}
\lim_{n\to\infty} F\left(x+\frac{1}{n}\right) &= \lim_{n\to\infty} \mu\left(\left(-\infty, x+\frac{1}{n}\right]\right) \\
&= \mu\left(\lim_{n\to\infty}\left(-\infty, x+\frac{1}{n}\right]\right) \\
&= \mu\left((-\infty, x]\right) = F(x).
\end{aligned}$$

Now, from F being non-decreasing function, $\limsup_{y\uparrow x} F(y) \leq F\left(x+\frac{1}{n}\right)$ for any $n \in \mathbb{N}$, and hence $\limsup_{y\downarrow x} F(y) \leq \lim_{n\to\infty} F\left(x+\frac{1}{n}\right) = F(x)$. Also for all $y > x$, $F(y) = \mu((-\infty, y]) \leq \mu((-\infty, x]) = F(x)$, and hence $\liminf_{y\downarrow x} F(y) \geq F(x)$. And hence

$$\lim_{y\downarrow x} F(y) = \liminf_{y\downarrow x} F(y) = \limsup_{y\downarrow x} F(y) = F(x),$$

and hence the right limit $\lim_{y\downarrow x} F(y)$ exists and equals $F(x)$.

8. Let $f : \Omega \to S$. Show that, for arbitrary subsets $A, A_1, A_2, \ldots$ of S,

(a) $f^{-1}(A^{\complement}) = (f^{-1}(A))^{\complement}$

(b) $f^{-1}(\cup_n A_n) = \bigcup_n f^{-1}(A_n)$ and

(c) $f^{-1}(\cap_n A_n) = \bigcap_n f^{-1}(A_n)$.

(The last two identities actually hold also for uncountable unions and intersections). Let $\mathcal{A}$ be a σ-field over S. Prove that the collection $f^{-1}(\mathcal{A}) = \{f^{-1}(A), A \in \mathcal{A}\}$ of subsets of Ω is a σ-field over Ω (in fact, the smallest σ-field on Ω that makes f measurable).

Points: 10 pts = 2 + 2 + 2 + 4.

Solution.

Note that $\omega \in f^{-1}(A)$ if and only if $f(\omega) \in A$.

(a)

$\omega \in f^{-1}(A^{\complement})$ and $\omega \in (f^{-1}(A))^{\complement}$ are equivalent as

$$\begin{aligned}
\omega \in f^{-1}(A^{\complement}) &\iff f(\omega) \in A^{\complement} \\
&\iff f(\omega) \notin A \\
&\iff \omega \notin f^{-1}(A) \\
&\iff \omega \in (f^{-1}(A))^{\complement}.
\end{aligned}$$

And hence $f^{-1}(A^{\complement}) = (f^{-1}(A))^{\complement}$.

(b)

$\omega \in f^{-1}\left(\bigcup_n A_n\right)$ and $\omega \in \bigcup_n f^{-1}(A_n)$ are equivalent as

$$\begin{aligned}
\omega \in f^{-1}\left(\bigcup_n A_n\right) &\iff f(\omega) \in \bigcup_n A_n \\
&\iff \text{there exists } n \text{ such that } f(\omega) \in A_n \\
&\iff \text{there exists } n \text{ such that } \omega \in f^{-1}(A_n) \\
&\iff \omega \in \bigcup_n f^{-1}(A_n).
\end{aligned}$$

And hence $f^{-1}\left(\bigcup_n A_n\right) = \bigcup_n f^{-1}(A_n)$.
(c)
$\omega \in f^{-1}\left(\bigcap_n A_n\right)$ and $\omega \in \bigcap_n f^{-1}(A_n)$ are equivalent as

$$\begin{aligned}
\omega \in f^{-1}\left(\bigcap_n A_n\right) &\iff f(\omega) \in \bigcap_n A_n \\
&\iff \text{for all } n,\ f(\omega) \in A_n \\
&\iff \text{for all } n,\ \omega \in f^{-1}(A_n) \\
&\iff \omega \in \bigcap_n f^{-1}(A_n).
\end{aligned}$$

And hence $f^{-1}\left(\bigcap_n A_n\right) = \bigcap_n f^{-1}(A_n)$.
(d)
We first show that $f^{-1}(\mathcal{A}) = \{f^{-1}(A), A \in \mathcal{A}\}$ is a σ-field over Ω. First, since $\mathcal{A}$ is a σ-field over S, $S \in \mathcal{A}$, and hence $\Omega = f^{-1}(S) \in f^{-1}(\mathcal{A})$. Second, for any $f^{-1}(A) \in f^{-1}(\mathcal{A})$, $\mathcal{A}$ being a σ-field and $A \in \mathcal{A}$ implies $A^{\complement} \in \mathcal{A}$, and hence $(f^{-1}(A))^{\complement} = f^{-1}(A^{\complement}) \in f^{-1}(\mathcal{A})$. Third, if $\{f^{-1}(A_n)\}_{n=1}^{\infty} \subset f^{-1}(\mathcal{A})$, then $\mathcal{A}$ being a σ-field and $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ implies $\bigcup_n A_n \in \mathcal{A}$, which implies $\bigcup_n f^{-1}(A_n) = f^{-1}(\bigcup_n A_n) \in f^{-1}(\mathcal{A})$. Hence $f^{-1}(\mathcal{A})$ is a σ-field.
Also, let $\mathcal{F}$ be the σ-field satisfying f to be measurable. Then for all $A \in \mathcal{A}$, $f^{-1}(A) \in \mathcal{F}$ from measurability of f, and hence $f^{-1}(\mathcal{A}) \subset \mathcal{F}$. Hence $f^{-1}(\mathcal{A})$ is indeed the smallest σ-field on Ω that makes f measurable.

9. Let $(\Omega, \mathcal{F})$ be a measurable space and $\{f_n\}_{n=1,2,\ldots}$ be a sequence of real valued measurable function on Ω. Show that the set $\{\omega \in \Omega : \lim_n f_n(\omega) \text{ exists}\}$ is measurable (i.e. it belongs to $\mathcal{F}$).

Points: 10 pts.

Solution.

Note that if $\{f_n\}_{n=1}^{\infty}$ are measurable functions, then for all n, $\inf_{k\geq n}\{f_k\}$ is a measurable function, and hence $\liminf_n f_n = \sup_n \inf_{k\geq n}\{f_k\}$ is a measurable function as well. Similarly, $\limsup_n f_n = \inf_n \sup_{k\geq n}\{f_k\}$ is a measurable function as well. Now, note that

$$\begin{aligned}\{\omega \in \Omega : \lim_n f_n(\omega) \text{ exists}\} &= \{\omega \in \Omega : \liminf_n f_n(\omega) = \limsup_n f_n(\omega)\} \\ &= (\limsup_n f_n - \liminf_n f_n)^{-1}(\{0\}).\end{aligned}$$

Since both $\liminf_n f_n$ and $\limsup_n f_n$ are measurable functions, $\limsup_n f_n - \liminf_n f_n$ is a measurable function as well. And $\{0\}$ is a Borel set, and hence $(\limsup_n f_n - \liminf_n f_n)^{-1}(\{0\})$ is a measurable set in Ω.

10. (**The induced measure is a measure**) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $(S, \mathcal{A})$ a measurable space and $f : \Omega \to S$ a measurable function. Show that the measure induced by f and μ, i.e. the function ν over $\mathcal{A}$ given by

$$A \mapsto \mu(f^{-1}(A)), \quad A \in \mathcal{A},$$

is a measure. Show by example that ν need not be σ-finite if μ is σ-finite.

Points: 10 pts.

Solution.

We check whether ν is a measure. First, for any $A \in \mathcal{A}$, $\nu(A) = \mu(f^{-1}(A)) \geq 0$.Second, since $f^{-1}(\emptyset) = \emptyset$, $\nu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$. Third, let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ be disjoint sets in $\mathcal{A}$, then since

$$f^{-1}(A_m) \cap f^{-1}(A_n) = f^{-1}(A_m \cap A_n) = f^{-1}(\emptyset) = \emptyset,$$

$\{f^{-1}(A_n)\}_{n=1}^{\infty}$ are disjoint sets in $\mathcal{F}$ as well. Hence by using countable additivity of μ,

$$\begin{aligned}\nu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right)\right) = \mu\left(\bigcup_{n=1}^{\infty} f^{-1}(A_n)\right) \\ &= \sum_{n=1}^{\infty} \mu(f^{-1}(A_n)) = \sum_{n=1}^{\infty} \nu(A_n).\end{aligned}$$

Hence countable additivity holds for ν as well. And hence ν is a measure.

Let $f : \mathbb{N} \to \{0\}$ be defined as $f(x) = 0$ for all $x \in \mathbb{N}$, $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ be such that μ is a counting measure on $\mathbb{N}$, and ν be the induced measure on $(\{0\}, 2^{\{0\}})$. Then $\{\{n\}\}_{n=1}^{\infty}$ is a countable subset of $\mathbb{N}$ such that $\bigcup_n \{n\} = \mathbb{N}$ and $\mu(\{n\}) = 1 < \infty$, hence μ is a σ-finite measure. However,

$$\nu(\{0\}) = \mu(f^{-1}(\{0\})) = \mu(\mathbb{N}) = \infty,$$

and hence ν has an infinite mass on a singleton set $\{0\}$. Then for any countable subset $A_n \subset \{0\}$ with $\bigcup_n A_n = \{0\}$, there exists $A_n \supset \{0\}$, and $\nu(A_n) \geq \nu(\{0\}) = \infty$. Hence ν cannot be σ-finite.