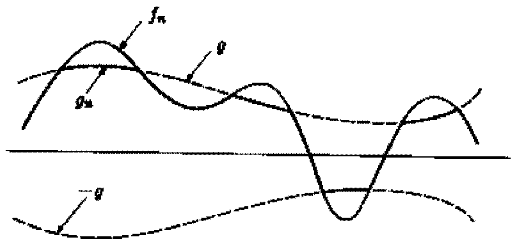

Figure 10.2

Geometrically, the function g_n is obtained from f_n by cutting off the graph of f_n from above by g and from below by $-g$, as shown by the example in Fig. 10.2. Then $|g_n(x)| \le g(x)$ almost everywhere on I, and it is easy to verify that $g_n \to f$ almost everywhere on I. Therefore, by the Lebesgue dominated convergence theorem, $f \in L(I)$.

10.12 LEBESGUE INTEGRALS ON UNBOUNDED INTERVALS AS LIMITS OF INTEGRALS ON BOUNDED INTERVALS

Theorem 10.31. *Let f be defined on the half-infinite interval $I = [a, +\infty)$. Assume that f is Lebesgue-integrable on the compact interval $[a, b]$ for each $b \ge a$, and that there is a positive constant M such that*

$$\int_a^b |f| \le M \qquad \text{for all } b \ge a. \tag{20}$$

Then $f \in L(I)$, the limit $\lim_{b\to+\infty} \int_a^b f$ exists, and

$$\int_a^{+\infty} f = \lim_{b\to+\infty} \int_a^b f. \tag{21}$$

Proof. Let $\{b_n\}$ be any increasing sequence of real numbers with $b_n \ge a$ such that $\lim_{n\to\infty} b_n = +\infty$. Define a sequence $\{f_n\}$ on I as follows:

$$f_n(x) = \begin{cases} f(x) & \text{if } a \le x \le b_n, \\ 0 & \text{otherwise.} \end{cases}$$

Each $f_n \in L(I)$ (by Theorem 10.18) and $f_n \to f$ on I. Hence, $|f_n| \to |f|$ on I. But $|f_n|$ is increasing and, by (20), the sequence $\{\int_I |f_n|\}$ is bounded above by M. Therefore $\lim_{n\to\infty} \int_I |f_n|$ exists. By the Levi theorem, the limit function $|f| \in L(I)$. Now each $|f_n| \le |f|$ and $f_n \to f$ on I, so by the Lebesgue dominated convergence theorem, $f \in L(I)$ and $\lim_{n\to\infty} \int_I f_n = \int_I f$. Therefore

$$\lim_{n\to\infty} \int_a^{b_n} f = \int_a^{+\infty} f$$

for all sequences $\{b_n\}$ which increase to $+\infty$. This completes the proof.

There is, of course, a corresponding theorem for the interval $(-\infty, a]$ which concludes that

$$\int_{-\infty}^a f = \lim_{c\to-\infty} \int_c^a f,$$

provided that $\int_c^a |f| \le M$ for all $c \le a$. If $\int_c^b |f| \le M$ for all real c and b with $c \le b$, the two theorems together show that $f \in L(\mathbf{R})$ and that

$$\int_{-\infty}^{+\infty} f = \lim_{c\to-\infty} \int_c^a f + \lim_{b\to+\infty} \int_a^b f.$$

Example 1. Let $f(x) = 1/(1 + x^2)$ for all x in $\mathbf{R}$. We shall prove that $f \in L(\mathbf{R})$ and that $\int_{\mathbf{R}} f = \pi$. Now f is nonnegative, and if $c \le b$ we have

$$\int_c^b f = \int_c^b \frac{dx}{1 + x^2} = \arctan b - \arctan c \le \pi.$$

Therefore, $f \in L(\mathbf{R})$ and

$$\int_{-\infty}^{+\infty} f = \lim_{c\to-\infty} \int_c^0 \frac{dx}{1 + x^2} + \lim_{b\to+\infty} \int_0^b \frac{dx}{1 + x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Example 2. In this example the limit on the right of (21) exists but $f \notin L(I)$. Let $I = [0, +\infty)$ and define f on I as follows:

$$f(x) = \frac{(-1)^n}{n} \qquad \text{if } n - 1 \le x < n, \quad \text{for } n = 1, 2, \ldots$$

If $b > 0$, let $m = [b]$, the greatest integer $\le b$. Then

$$\int_0^b f = \int_0^m f + \int_m^b f = \sum_{n=1}^m \frac{(-1)^n}{n} + \frac{(b - m)(-1)^{m+1}}{m + 1}.$$

As $b \to +\infty$ the last term $\to 0$, and we find

$$\lim_{b\to+\infty} \int_0^b f = \sum_{n=1}^\infty \frac{(-1)^n}{n} = -\log 2.$$

Now we assume $f \in L(I)$ and obtain a contradiction. Let f_n be defined by

$$f_n(x) = \begin{cases} |f(x)| & \text{for } 0 \le x \le n, \\ 0 & \text{for } x > n. \end{cases}$$

Then $\{f_n\}$ increases and $f_n(x) \to |f(x)|$ everywhere on I. Since $f \in L(I)$ we also have $|f| \in L(I)$. But $|f_n(x)| \le |f(x)|$ everywhere on I so by the Lebesgue dominated convergence theorem the sequence $\{\int_I f_n\}$ converges. But this is a contradiction since

$$\int_I f_n = \int_0^n |f| = \sum_{k=1}^n \frac{1}{k} \to +\infty \qquad \text{as } n \to \infty.$$

10.13 IMPROPER RIEMANN INTEGRALS

Definition 10.32. *If f is Riemann-integrable on $[a, b]$ for every $b \geq a$, and if the limit*

$$\lim_{b \to +\infty} \int_a^b f(x)\, dx \quad \textit{exists,}$$

then f is said to be improper Riemann-integrable on $[a, +\infty)$ and the improper Riemann integral of f, denoted by $\int_a^{+\infty} f(x)\, dx$ or $\int_a^{\infty} f(x)\, dx$, is defined by the equation

$$\int_a^{+\infty} f(x)\, dx = \lim_{b \to +\infty} \int_a^b f(x)\, dx.$$

In Example 2 of the foregoing section the improper Riemann integral $\int_0^{+\infty} f(x)\, dx$ exists but f is not Lebesgue-integrable on $[0, +\infty)$. That example should be contrasted with the following theorem.

Theorem 10.33. *Assume f is Riemann-integrable on $[a, b]$ for every $b \geq a$, and assume there is a positive constant M such that*

$$\int_a^b |f(x)|\, dx \leq M \qquad \textit{for every } b \geq a. \tag{22}$$

Then both f and $|f|$ are improper Riemann-integrable on $[a, +\infty)$. Also, f is Lebesgue-integrable on $[a, +\infty)$ and the Lebesgue integral of f is equal to the improper Riemann integral of f.

Proof. Let $F(b) = \int_a^b |f(x)|\, dx$. Then F is an increasing function which is bounded above by M, so $\lim_{b \to +\infty} F(b)$ exists. Therefore $|f|$ is improper Riemann-integrable on $[a, +\infty)$. Since

$$0 \leq |f(x)| - f(x) \leq 2|f(x)|,$$

the limit

$$\lim_{b \to +\infty} \int_a^b \{|f(x)| - f(x)\}\, dx$$

also exists; hence the limit $\lim_{b \to +\infty} \int_a^b f(x)\, dx$ exists. This proves that f is improper Riemann-integrable on $[a, +\infty)$. Now we use inequality (22), along with Theorem 10.31, to deduce that f is Lebesgue-integrable on $[a, +\infty)$ and that the Lebesgue integral of f is equal to the improper Riemann integral of f.

NOTE. There are corresponding results for improper Riemann integrals of the form

$$\int_{-\infty}^b f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx,$$

$$\int_a^c f(x)\, dx = \lim_{b \to c-} \int_a^b f(x)\, dx,$$

and

$$\int_c^b f(x)\, dx = \lim_{a \to c+} \int_a^b f(x)\, dx,$$

which the reader can formulate for himself.

If both integrals $\int_{-\infty}^a f(x)\, dx$ and $\int_a^{+\infty} f(x)\, dx$ exist, we say that the integral $\int_{-\infty}^{+\infty} f(x)\, dx$ exists, and its value is defined to be their sum,

$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_{-\infty}^{a} f(x)\, dx + \int_a^{+\infty} f(x)\, dx.$$

If the integral $\int_{-\infty}^{+\infty} f(x)\, dx$ exists, its value is also equal to the symmetric limit

$$\lim_{b \to +\infty} \int_{-b}^b f(x)\, dx.$$

However, it is important to realize that the symmetric limit might exist even when $\int_{-\infty}^{+\infty} f(x)\, dx$ does not exist (for example, take $f(x) = x$ for all x). In this case the symmetric limit is called the *Cauchy principal value* of $\int_{-\infty}^{+\infty} f(x)\, dx$. Thus $\int_{-\infty}^{+\infty} x\, dx$ has Cauchy principal value 0, but the integral does not exist.

Example 1. Let $f(x) = e^{-x}x^{y-1}$, where y is a fixed real number. Since $e^{-x/2}x^{y-1} \to 0$ as $x \to +\infty$, there is a constant M such that $e^{-x/2}x^{y-1} \leq M$ for all $x \geq 1$. Then $e^{-x}x^{y-1} \leq Me^{-x/2}$, so

$$\int_1^b |f(x)|\, dx \leq M \int_0^b e^{-x/2}\, dx = 2M(1 - e^{-b/2}) < 2M.$$

Hence the integral $\int_1^{+\infty} e^{-x}x^{y-1}\, dx$ exists for every real y, both as an improper Riemann integral and as a Lebesgue integral.

Example 2. *The Gamma function integral.* Adding the integral of Example 1 to the integral $\int_0^1 e^{-x}x^{y-1}\, dx$ of Example 2 of Section 10.9, we find that the Lebesgue integral

$$\Gamma(y) = \int_0^{+\infty} e^{-x}x^{y-1}\, dx$$

exists for each real $y > 0$. The function Γ so defined is called the *Gamma function.* Example 4 below shows its relation to the Riemann zeta function.

NOTE. Many of the theorems in Chapter 7 concerning Riemann integrals can be converted into theorems on improper Riemann integrals. To illustrate the straightforward manner in which some of these extensions can be made, consider the formula for integration by parts:

$$\int_a^b f(x)g'(x)\, dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)\, dx.$$

Since b appears in three terms of this equation, there are three limits to consider as $b \to +\infty$. If two of these limits exist, the third also exists and we get the formula

$$\int_a^\infty f(x)g'(x)\,dx = \lim_{b\to+\infty} f(b)g(b) - f(a)g(a) - \int_a^\infty g(x)f'(x)\,dx.$$

Other theorems on Riemann integrals can be extended in much the same way to improper Riemann integrals. However, it is not necessary to develop the details of these extensions any further, since in any particular example, it suffices to apply the required theorem to a compact interval $[a, b]$ and then let $b \to +\infty$.

Example 3. *The functional equation* $\Gamma(y+1) = y\Gamma(y)$. If $0 < a < b$, integration by parts gives

$$\int_a^b e^{-x}x^y\,dx = a^y e^{-a} - b^y e^{-b} + y\int_a^b e^{-x}x^{y-1}\,dx.$$

Letting $a \to 0+$ and $b \to +\infty$, we find $\Gamma(y+1) = y\Gamma(y)$.

Example 4. *Integral representation for the Riemann zeta function.* The Riemann zeta function ζ is defined for $s > 1$ by the equation

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}.$$

This example shows how the Levi convergence theorem for series can be used to derive an integral representation,

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx.$$

The integral exists as a Lebesgue integral.

In the integral for $\Gamma(s)$ we make the change of variable $t = nx$, $n > 0$, to obtain

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt = n^s\int_0^\infty e^{-nx}x^{s-1}\,dx.$$

Hence, if $s > 0$, we have

$$n^{-s}\Gamma(s) = \int_0^\infty e^{-nx}x^{s-1}\,dx.$$

If $s > 1$, the series $\sum_{n=1}^\infty n^{-s}$ converges, so we have

$$\zeta(s)\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty e^{-nx}x^{s-1}\,dx,$$

the series on the right being convergent. Since the integrand is nonnegative, Levi's convergence theorem (Theorem 10.25) tells us that the series $\sum_{n=1}^\infty e^{-nx}x^{s-1}$ converges almost everywhere to a sum function which is Lebesgue-integrable on $[0, +\infty)$ and that

$$\zeta(s)\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty e^{-nx}x^{s-1}\,dx = \int_0^\infty \sum_{n=1}^\infty e^{-nx}x^{s-1}\,dx.$$

But if $x > 0$, we have $0 < e^{-x} < 1$ and hence,

$$\sum_{n=1}^\infty e^{-nx} = \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{e^x - 1},$$

the series being a geometric series. Therefore we have

$$\sum_{n=1}^\infty e^{-nx}x^{s-1} = \frac{x^{s-1}}{e^x - 1}$$

almost everywhere on $[0, +\infty)$, in fact everywhere except at 0, so

$$\zeta(s)\Gamma(s) = \int_0^\infty \sum_{n=1}^\infty e^{-nx}x^{s-1}\,dx = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx.$$

10.14 MEASURABLE FUNCTIONS

Every function f which is Lebesgue-integrable on an interval I is the limit, almost everywhere on I, of a certain sequence of step functions. However, the converse is not true. For example, the constant function $f = 1$ is a limit of step functions on the real line $\mathbf{R}$, but this function is not in $L(\mathbf{R})$. Therefore, the class of functions which are limits of step functions is larger than the class of Lebesgue-integrable functions. The functions in this larger class are called *measurable functions.*

Definition 10.34. *A function f defined on I is called measurable on I, and we write $f \in M(I)$, if there exists a sequence of step functions $\{s_n\}$ on I such that*

$$\lim_{n\to\infty} s_n(x) = f(x) \quad \textit{almost everywhere on } I.$$

NOTE. If f is measurable on I then f is measurable on every subinterval of I.

As already noted, every function in $L(I)$ is measurable on I, but the converse is not true. The next theorem provides a partial converse.

Theorem 10.35. *If $f \in M(I)$ and if $|f(x)| \le g(x)$ almost everywhere on I for some nonnegative g in $L(I)$, then $f \in L(I)$.*

Proof. There is a sequence of step functions $\{s_n\}$ such that $s_n(x) \to f(x)$ almost everywhere on I. Now apply Theorem 10.30 to deduce that $f \in L(I)$.

Corollary 1. *If $f \in M(I)$ and $|f| \in L(I)$, then $f \in L(I)$.*

Corollary 2. *If f is measurable and bounded on a bounded interval I, then $f \in L(I)$.*

Further properties of measurable functions are given in the next theorem.

Theorem 10.36. *Let φ be a real-valued function continuous on $\mathbf{R}^2$. If $f \in M(I)$ and $g \in M(I)$, define h on I by the equation*

$$h(x) = \varphi[f(x), g(x)].$$