Nonmeasurable Sets

There exist in $(0,1]$ *sets that lie outside* $\mathscr{B}$. For the construction (due to Vitali) it is convenient to use addition modulo 1 in $(0,1]$. For $x, y \in (0,1]$ take $x \oplus y$ to be $x+y$ or $x+y-1$ according as $x+y$ lies in $(0,1]$ or not.† Put $A \oplus x = [a \oplus x\colon a \in A]$.

Let $\mathscr{L}$ be the class of Borel sets A such that $A \oplus x$ is a Borel set and $\lambda(A \oplus x) = \lambda(A)$. Then $\mathscr{L}$ is a λ-system containing the intervals, and so $\mathscr{B} \subset \mathscr{L}$ by the π-λ theorem. Thus $A \in \mathscr{B}$ implies that $A \oplus x \in \mathscr{B}$ and $\lambda(A \oplus x) = \lambda(A)$. In this sense, λ is translation-invariant.

Define x and y to be equivalent $(x \sim y)$ if $x \oplus r = y$ for some rational r in $(0,1]$. Let H be a subset of $(0,1]$ consisting of exactly one representative point from each equivalence class; such a set exists under the assumption of the axiom of choice [A8]. Consider now the countably many sets $H \oplus r$ for rational r

These sets are disjoint, because no two distinct points of H are equivalent. (If $H \oplus r_1$ and $H \oplus r_2$ share the point $h_1 \oplus r_1 = h_2 \oplus r_2$, then $h_1 \sim h_2$; this is impossible unless $h_1 = h_2$, in which case $r_1 = r_2$.) Each point of $(0,1]$ lies in one of these sets, because H has a representative from each equivalence class. (If $x \sim h \in H$, then $x = h \oplus r \in H \oplus r$ for some rational r.) Thus $(0,1] = \bigcup_r (H \oplus r)$, a countable disjoint union.

If H were in $\mathscr{B}$, it would follow that $\lambda(0,1] = \sum_r \lambda(H \oplus r)$. This is impossible: If the value common to the $\lambda(H \oplus r)$ is 0, it leads to $1 = 0$; if the common value is positive, it leads to a convergent infinite series of identical positive terms ($a + a + \cdots < \infty$ and $a > 0$). Thus H lies outside $\mathscr{B}$. ■