36-752 Advanced Probability Overview	Spring 2018

3. Limit Theorems and the Standard Machinery

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Associated reading: Sec 1.6 and 2.2 of Ash and Doléans-Dade; Sec 1.5 and A.4 of Durrett.

1 Basic Limit Theorems and Applications

One of the famous limit theorems is the following.

Theorem 1 (Fatou's lemma). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions. Then*

$$\int \liminf_n f_n d\mu \le \liminf_n \int f_n d\mu.$$

Proof: Let $f(\omega) = \liminf_{n\to\infty} f_n(\omega)$. Because

$$\int f d\mu = \sup_{\text{finite simple } \phi \le f} \int \phi d\mu,$$

we need only prove that, for every finite simple $\phi \le f$,

$$\int \phi d\mu \le \liminf_{n\to\infty} \int f_n d\mu.$$

Let $\phi \le f$ be finite and simple, and let $\epsilon > 0$. For each n, define

$$A_n = \{\omega \in \Omega : f_k(\omega) \ge (1-\epsilon)\phi(\omega), \text{ for all } k \ge n\}.$$

Since $(1-\epsilon)\phi(\omega) \le f(\omega)$ for all ω with strict inequality wherever either side is positive, $\bigcup_{n=1}^{\infty} A_n = \Omega$ and $A_n \subseteq A_{n+1}$ for all n.

$$\int f_n d\mu \ge \int_{A_n} f_n d\mu \ge (1-\epsilon)\int_{A_n} \phi d\mu. \tag{1}$$

Let the canonical representation of ϕ be $\sum_{i=1}^m c_i I_{C_i}$. Then, for all n.

$$\int_{A_n} \phi d\mu = \sum_{i=1}^m c_i \mu(C_i \cap A_n).$$

Because the A_n's form an increasing sequence whose union is Ω, $\lim_{n\to\infty}\mu(C_i\cap A_n)=\mu(C_i)$ for all i. Taking the $\liminf_n$ of both sides of Equation (1) yields

$$\liminf_n \int f_n d\mu \geq (1-\epsilon)\sum_{i=1}^{m} c_i\mu(C_i) = (1-\epsilon)\int \phi d\mu.$$

Since this is true for every $\epsilon > 0$,

$$\liminf_{n\to\infty} \int f_n d\mu \geq \int \phi d\mu.$$

■

The first of the two most useful limit theorems is the following.

Theorem 2 (Monotone convergence theorem). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable nonnegative functions, and let f be a measurable function such that $f_n \leq f$ and $\lim_{n\to\infty} f_n = f$. Then,*

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Proof: Since $f_n \leq f$ for all n, $\int f_n d\mu \leq \int f d\mu$ for all n. Hence

$$\liminf_{n\to\infty} \int f_n d\mu \leq \limsup_{n\to\infty} \int f_n d\mu \leq \int f d\mu.$$

By Fatou's lemma, $\int f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$. ■

Exercise 3. *Why is it called the "monotone" convergence theorem?*

Exercise 4. *Suppose that f_n is integrable for each n and $\sup_n \int f_n d\mu < \infty$. Show that, if $f_n \uparrow f$, then f is integrable and $\int f_n d\mu \to \int f d\mu$.*

Exercise 5. *Assume the sequence of functions f_n is defined on a measure space $(\Omega, \mathcal{F}, \mu)$ such that $\mu(\Omega) < \infty$. Further, suppose that the f_n are uniformly bounded and that $f_n \to f$ uniformly. Show that $\int f_n d\mu \to \int f d\mu$.*

We are now in a position to prove properties such as linearity and change of variable formula, using the "Standard Machinery".

2 The Standard Machinery

2.1 Linearity of Integral

Theorem 6 (Linearity of Integrals). *If $\int f d\mu$ and $\int g d\mu$ are defined and they are not both infinite and of opposite signs, then $\int [f+g] d\mu = \int f d\mu + \int g d\mu$.*

Proof: If $f, g \geq 0$, then by monotone approximation, there exist sequences of nonnegative simple functions $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ such that $f_n \uparrow f$ and $g_n \uparrow g$. Then $(f_n + g_n) \uparrow (f+g)$ and $\int [f_n + g_n] d\mu = \int f_n d\mu + \int g_n d\mu$ by linearity of integrals of simple functions. The result now follows from the monotone convergence theorem. For integrable f and g, note that $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$. What we just proved for nonnegative functions implies that

$$\begin{aligned}
&\int (f+g)^+ d\mu + \int f^- d\mu + \int g^- d\mu \\
&= \int [(f+g)^+ + f^- + g^-] d\mu \\
&= \int [(f+g)^- + f^+ + g^+] d\mu \\
&= \int (f+g)^- d\mu + \int f^+ d\mu + \int g^+ d\mu.
\end{aligned}$$

Rearranging the terms in the first and last expressions gives the desired result. If both f and g have infinite integral of the same sign, then it follows easily that $f+g$ has infinite integral of the same sign. Finally, if only one of f and g has infinite integral, it also follows easily that $f+g$ has infinite integral of the same sign. ∎

For proving theorems about integrals, there is a common sequence of steps that is often called the *standard machinery*. The standard machinery essentially entails the following steps: (1) prove the claim about the integral for non-negative simple functions; (2) use the monotone convergence theorem to show that the claim holds for non-negative measurable functions and (3) use the decomposition $f = f^+ - f^-$ to deal with general measurable functions. We illustrate this machinery in the next few results.

2.2 Change of variable

The first illustration is the measure-theoretic version of the change-of-variables formula.

Lemma 7 (Change of Variable). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $(S, \mathcal{A})$ be a measurable space. Let $f : \Omega \to S$ be a measurable function. Let ν be the measure induced on*

$(S, \mathcal{A})$ by f from μ. (See the Induced Measure Lemma.) Let $g : S \to I\!R$ be $\mathcal{A}/\mathcal{B}^1$ measurable. Then

$$\int g d\nu = \int g(f) d\mu, \tag{2}$$

if either integral exists.

Proof: First, assume that $g = I_A$ for some $A \in \mathcal{A}$. Then Equation (2) becomes $\nu(A) = \mu(f^{-1}(A))$, which is the definition of ν. Next, if g is a nonnegative simple function, then Equation (2) holds by linearity of integrals. If g is a nonnegative function, then use the monotone convergence theorem and a sequence of nonnegative simple functions converging to g from below to see that Equation (2) holds. Finally, for general g, Equation (2) holds if either g^+ or g^- is integrable. ■

Lemma 7 has a widely-used corollary.

Corollary 8 (Law of the unconscious statistician). *If $X : \Omega \to S$ is a random quantity with distribution μ_X and if $f : S \to I\!R$ is measurable, then $\mathrm{E}[f(X)] = \int f d\mu_X$.*

2.3 Density Functions

Another useful application of monotone convergence is the following.

Theorem 9. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $f : \Omega \to \overline{I\!R}^{+0}$ be measurable. Then $\nu(A) = \int_A f d\mu$ is a measure on $(\Omega, \mathcal{F})$.*

Exercise 10. *Prove Theorem 9.*

If μ is σ-finite and if f is finite a.e. $[\mu]$, then ν in Theorem 9 is σ-finite.

What goes wrong with the conclusion to Theorem 9 if f is integrable but not necessarily nonnegative? If f can take negative values then $\nu(A) = \int_A f d\mu$ might be negative. Let $A = \{\omega : f(\omega) < 0\}$. Suppose that $\mu(A) > 0$. Write $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n = \{\omega : f(\omega) < -1/n\}$. If $\mu(A) > 0$, then there exists n such that $\mu(A_n) > 0$. (This argument is used often in proving probability results.) Then

$$-\nu(A) = \int I_A(-f) d\mu \geq \int I_{A_n}(-f) d\mu \geq \frac{1}{n}\mu(A_n) > 0.$$

Here is another application of the standard machinery.

Theorem 11 (Density Function). *Assume the same conditions as Theorem 9. Integrals with respect to ν can be computed as $\int g d\nu = \int f g d\mu$, if either exists.*

Proof: We prove the result in four stages. First, assume that g is a indicator I_A of some set $A \in \mathcal{F}$. Then the definition of ν says that $\int g d\nu = \nu(A) = \int I_A f d\mu$. Second, assume that g is a nonnegative simple function. The result holds for g by linearity of integrals. Third, assume that g is nonnegative. Approximate g from below by nonnegative simple functions $\{g_n\}_{n=1}^{\infty}$. Then $\int g_n d\nu = \int g_n f d\mu$ for each n and the monotone convergence theorem says that the left side converges to $\int g d\nu$ and the right side converges to $\int g f d\mu$. Finally, if g is measurable, write $g = g^+ - g^-$ (the positive and negative parts). Then $\int g^+ d\nu = \int g^+ f d\mu$ and $\int g^- d\nu = \int g^- f d\mu$. We see that $\int g d\nu$ exists if and only if $\int g f d\mu$ exists, and if either exists they are equal. ■

Definition 12. *The function f in Theorem 9 is called the* density of ν with respect to μ*.*

Example 13 (Probability density functions). *Consider a continuous random variable X having a density f. That is,*

$$\Pr(X \le a) = \int_{-\infty}^{a} f(x)dx.$$

Then the distribution of X, defined by $\mu_X(B) = \Pr(X \in B)$ for $B \in \mathcal{B}^1$, satisfies

$$\mu_X(B) = \int_B f d\lambda,$$

where λ is Lebesgue measure. That is, the probability density functions of the usual continuous distributions that you learned about in earlier courses are also densities with respect to Lebesgue measure in the sense defined above.

Example 14 (Probability mass functions). *Consider a typical discrete random variable X with mass function f, i.e., $f(x) = \Pr(X = x)$ for all x. There are at most countably many x such that $f(x) > 0$. Let μ_X be the distribution of X. For each set B, we know that*

$$\mu_X(B) = \Pr(X \in B) = \sum_{x \in B} f(x).$$

The rightmost term in this equation is $\int f d\mu$, where μ is counting measure on the range space of X. So, f is the density of μ_X with respect to μ.

3 Additional Properties of Integrals

Here are some more useful properties of integrals.

Theorem 15. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let f and g be measurable extended real-valued functions.*

1. *If f is nonnegative and $\mu(\{\omega : f(\omega) > 0\}) > 0$, then $\int f d\mu > 0$.*

2. *If f and g are integrable and if $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{F}$, then $f = g$ a.e. $[\mu]$.*

3. *If μ is σ-finite and if $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{F}$, then $f = g$ a.e. $[\mu]$.*

4. *Let Π be a π-system that generates $\mathcal{F}$. Suppose that Ω is a finite or countable union of elements of Π. If f and g are integrable and if $\int_A f d\mu = \int_A g d\mu$ for all $A \in \Pi$, then $f = g$ a.e. $[\mu]$.*

Proof: 1. Let $A_c = \{\omega : f(\omega) > c\}$ for each $c \geq 0$. Because $\mu(A_0) > 0$ and $A_0 = \bigcup_{n=1}^{\infty} A_{1/n}$, then there exists n such that $\mu(A_{1/n}) > 0$ (by the limit result on measure of monotone sequence of sets). Since $f \geq f I_{A_{1/n}}$, we have $\int f d\mu \geq \int_{A_{1/n}} f d\mu$. But $(1/n) I_{A_{1/n}}$ is a simple function that is $\leq f I_{A_{1/n}}$ and $\int (1/n) I_{A_{1/n}} d\mu = n^{-1}\mu(A_{1/n}) > 0$. It follows that $\int f d\mu > 0$.

2. This will appear on a homework assignment.

3. First, assume that f and g are real-valued. Let $\{A_n\}_{n=1}^{\infty}$ be disjoint elements of $\mathcal{F}$ such that $\mu(A_n) < \infty$ and $\bigcup_{n=1}^{\infty} A_n = \Omega$. Let $B_m = \{\omega : |f(\omega)| < m, |g(\omega)| < m\}$ for each integer m. For each pair (n, m), $f I_{A_n \cap B_m}$ and $g I_{A_n \cap B_m}$ satisfy the conditions of the previous part, so $f I_{A_n \cap B_m} = g I_{A_n \cap B_m}$ a.e. $[\mu]$. Let $C = \{\omega : f(\omega) \neq g(\omega)\}$. Since

$$C = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} [C \cap B_m \cap A_n],$$

and each $\mu(C \cap B_m \cap A_n) = 0$, it follows that $\mu(C) = 0$.

Next, suppose that f and/or g is extended real-valued. Let $E = \{f = \infty\} \Delta \{g = \infty\}$, the set where one function is ∞ but the other is not. If $\mu(E) > 0$, then there is a subset A of E such that $0 < \mu(A) < \infty$ and one of the functions is bounded above on A while the other is infinite. This contradicts $\int_A f d\mu = \int_A g d\mu$. A similar result holds for $-\infty$.

4. Define $\nu_1^+(A) = \int_A f^+ d\mu$, $\nu_2^+(A) = \int_A g^+ d\mu$, $\nu_1^-(A) = \int_A f^- d\mu$, and $\nu_2^-(A) = \int_A g^- d\mu$. These are all finite measures according to Theorem 9. The additional condition implies that they are all σ-finite on Π. The equality of the integrals implies that $\nu_1^+ + \nu_2^- = \nu_1^- + \nu_2^+$ for all sets in Π. The uniqueness theorem implies that $\nu_1^+ + \nu_2^- = \nu_1^- + \nu_2^+$ for all sets in $\mathcal{F}$. Hence, the condition of part 2 hold and the result is proven. □

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The condition about unions in part 4 of the above theorem holds for the π-systems of the form $\{(a, b]\}$ or $\{(-\infty, b]\}$.

Corollary 16. *If μ is σ-finite and ν is related to μ as in Theorem 9, then the density of ν with respect to μ is unique, a.e. $[\mu]$.*

There is an interesting characterization of σ-finite measures in terms of integrals.

Theorem 17. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then μ is σ-finite if and only if there exists a strictly positive integrable function.*

Exercise 18. *Prove Theorem 17.*

In general, given another measure ν on $(\Omega, \mathcal{F})$, can we find such a density function? Of course, the existence of such a density function requires some special relationship between ν and the "base measure" μ.

4 Other Limit Theorems

The other major limit theorem is the following.

Theorem 19 (Dominated convergence theorem). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions, and let f and g be measurable functions such that $f_n \to f$ a.e. $[\mu]$, $|f_n| \leq g$ a.e. $[\mu]$, and $\int g d\mu < \infty$. Then,*

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Proof: We have $-g \leq f_n \leq g$ a.e. $[\mu]$, hence

$$\begin{aligned}
g + f_n &\geq 0, \quad \text{a.e. } [\mu], \\
g - f_n &\geq 0, \quad \text{a.e. } [\mu], \\
\lim_{n\to\infty} [g + f_n] &= g + f \quad \text{a.e. } [\mu], \\
\lim_{n\to\infty} [g - f_n] &= g - f \quad \text{a.e. } [\mu].
\end{aligned}$$

It follows from Fatou's lemma and Theorem 6 that

$$\begin{aligned}
\int [g + f] d\mu &\leq \liminf_{n\to\infty} \int [g + f_n] d\mu \\
&= \int g d\mu + \liminf_{n\to\infty} \int f_n d\mu, \\
\int f d\mu &\leq \liminf_{n\to\infty} \int f_n d\mu.
\end{aligned}$$

Similarly, it follows that

$$\begin{aligned}\int [g - f] d\mu &\leq \liminf_{n\to\infty} \int [g - f_n] d\mu \\ &= \int g d\mu - \limsup_{n\to\infty} \int f_n d\mu, \\ \int f d\mu &\geq \limsup_{n\to\infty} \int f_n d\mu.\end{aligned}$$

Together, these imply the conclusion of the theorem. ■

Example 20. *Let μ be a finite measure. Then limits and integrals can be interchanged whenever the functions in the sequence are uniformly bounded.*

An alternate version of the dominated convergence theorem is the following.

Proposition 21. *Let $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ be sequences of measurable functions such that $|f_n| \leq g_n$, a.e. $[\mu]$. Let f and g be measurable functions such that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$, a.e. $[\mu]$. Suppose that $\lim_{n\to\infty} \int g_n d\mu = \int g d\mu < \infty$. Then, $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$.*

The proof is the same as the proof of Theorem 19, except that g_n replaces g in the first three lines and wherever g appears with f_n and a limit is being taken.

For finite measure spaces (i.e. $(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) < \infty$), the minimal condition that guarantees convergence of integrals is *uniform integrability* .

Definition 22 (Uniform Integrability). *A sequence of integrable functions $\{f_n\}_{n=1}^{\infty}$ is* uniformly integrable (with respect to μ) *if $\lim_{c\to\infty} \sup_n \int_{\{\omega:|f_n(\omega)|>c\}} |f_n| d\mu = 0$.*

Theorem 23. *Let μ be a finite measure. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of integrable functions such that $\lim_{n\to\infty} f_n = f$ a.e. $[\mu]$. Suppose that $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable. Then $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$.*

If the f_n's in Theorem 23 are nonnegative and integrable and $f_n \to f$, then $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$ implies that $\{f_n\}_{n=1}^{\infty}$ are uniformly integrable. We will not use this result, however.

5 Absolute Continuity and R-N Derivative

Definition 24 (Absolute Continuity). *Let ν and μ be measures on the space $(\Omega, \mathcal{F})$. We say that $\nu \ll \mu$ (read ν* is absolutely continuous with respect to *μ) if for every $A \in \mathcal{F}$, $\mu(A) = 0$ implies $\nu(A) = 0$.*

That is, $\nu \ll \mu$ if and only if every measure 0 set under μ is also a measure 0 set under ν.

Example 25. *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space. Let* f *be a nonnegative function, and define* $\nu(A) = \int_A f d\mu$*. Then* ν *is a measure and* $\nu \ll \mu$*. If* $f < \infty$ *a.e.* $[\mu]$ *and if* μ *is* σ*-finite, then* ν *is* σ*-finite as well.*

Example 26. *Let* μ_1 *and* μ_2 *be measures on the same space. Let* $\mu = \mu_1 + \mu_2$*. Then* $\mu_i \ll \mu$ *for* $i = 1, 2$*.*

Absolute continuity can be interpreted as "continuity of measures".

Proposition 27. *Let* ν *and* μ *be measures on the space* $(\Omega, \mathcal{F})$*. Suppose that, for every* $\epsilon > 0$*, there exists* δ *such that for every* $A \in \mathcal{F}$*,* $\mu(A) < \delta$ *implies* $\nu(A) < \epsilon$*. Then* $\nu \ll \mu$*. Conversely, if* $\nu \ll \mu$ *and* ν*,* μ *are finite, then for every* $\epsilon > 0$*, there exists* δ *such that for every* $A \in \mathcal{F}$*,* $\mu(A) < \delta$ *implies* $\nu(A) < \epsilon$*.*

A concept related to absolute continuity is singularity.

Definition 28 (Mutually singular measures). *Two measures* μ *and* ν *on the same space* $(\Omega, \mathcal{F})$ *are* (mutually) singular *(denoted* $\mu \perp \nu$*) if there exist disjoint sets* S_μ *and* S_ν *such that* $\mu(S_\mu^C) = \nu(S_\nu^C) = 0$*.*

Example 29. *Let* f *and* g *be nonnegative functions such that* $fg = 0$ *a.e.* $[\mu]$*. Define* $\nu_1(A) = \int_A f d\mu$ *and* $\nu_2(A) = \int_A g d\mu$*. Then* $\nu_1 \perp \nu_2$*.*

The main theoretical result on absolute continuity is the Radon-Nikodym theorem which says that, in the σ-finite case, all absolute continuity is of the type in Example 25.

Theorem 30 (Radon-Nikodym). *Let* μ *and* ν *be* σ*-finite measures on the space* $(\Omega, \mathcal{F})$*. Then* $\nu \ll \mu$ *if and only if there exists a nonnegative measurable* f *such that* $\nu(A) = \int_A f d\mu$ *for all* $A \in \mathcal{F}$*. The function* f *is unique a.e.* $[\mu]$*.*

One proof of this result is given at the end of this set of lecture notes. Another proof is given later after we introduce conditional expectation.

Definition 31 (R-N Derivative). *The function* f *in Theorem 30 is called a* Radon-Nikodym derivative of ν with respect to μ*. It is denoted* $d\nu/d\mu$*. Each such function is called a* version *of* $d\nu/d\mu$*.*

From the proof of Theorem 30 we know that if ν is not absolutely continuous with respect to μ, we can decompose it into an absolutely continuous part and a singular part.

Theorem 32 (Lebesgue Decomposition). *Let* ν *and* μ *be two* σ*-finite measures on* $(\Omega, \mathcal{F})$*. Then there exists two* σ*-finite measures* ν_0 *and* ν_1 *such that* $\nu = \nu_0 + \nu_1$*,* $\nu_0 \ll \mu$*,* $\nu_1 \perp \mu$*.*

The uniqueness of Radon-Nikodym derivatives is only a.e. $[\mu]$. If $f = d\nu/d\mu$, then every measurable function that equals f a.e. $[\mu]$ could also be called $d\nu/d\mu$. All of these functions are called *versions* of the Radon-Nikodym derivative.

Definition 33 (Equivalent Measures). *If $\mu \ll \nu$ and $\nu \ll \mu$, we say that μ and ν are* equivalent.

If μ and ν are equivalent, then

$$\frac{d\mu}{d\nu} = \frac{1}{\frac{d\nu}{d\mu}}.$$

If $\nu \ll \mu \ll \eta$, then the chain rule for R-N derivatives says

$$\frac{d\nu}{d\eta} = \frac{d\nu}{d\mu}\frac{d\mu}{d\eta}.$$

Radon-Nikodym Theorem

THEOREM 30. (RADON-NIKODYM) *Let μ and ν be σ-finite measures on the space $(\Omega, \mathcal{F})$. Then $\nu \ll \mu$ if and only if there exists a nonnegative measurable f such that $\nu(A) = \int_A f d\mu$ for all $A \in \mathcal{F}$. The function f is unique a.e. $[\mu]$.*

The proof of this result relies upon the theory of signed measures.

Definition 34 (Signed Measure). *Let $(\Omega, \mathcal{F})$ be a measurable space. Let $\eta : \mathcal{F} \to \overline{I\!R}$. We call η a* signed measure *if*

- $\eta(\varnothing) = 0$,
- *for every sequence $\{A_k\}_{k=1}^{\infty}$ of mutually disjoint elements of $\mathcal{F}$, $\eta(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \eta(A_k)$.*
- *η takes at most one of the two values $\pm\infty$.*

Example 35. *Let μ_1 and μ_2 be measures on the same space such that at most one of them is infinite. Then $\mu_1 - \mu_2$ is a signed measure.*

Example 36. *Let f be integrable with respect to μ, and define $\eta(A) = \int_A f d\mu$. Then f is a finite signed measure. If the integral of f is merely defined, but not finite, then $\int_A f d\mu$ is a signed measure.*

The nice thing about signed measures is that they divide up nicely into positive and negative parts just like measurable functions.

Theorem 37 (Hahn and Jordan decompositions). *Let η be a signed measure on $(\Omega, \mathcal{F})$. Then there exists a set A^+ such that every subset A of A^+ has $\eta(A) \geq 0$ and every subset B of A^{+C} has $\eta(B) \leq 0$. Also, there exist finite mutually singular measures η_+ and η_- such that $\eta = \eta_+ - \eta_-$.*

Proof: With out loss of generality, assume that η does not take the value ∞. As a result, $\sup_{A \in \mathcal{F}} \eta(A) < \infty$. Let $\alpha = \sup_{A \in \mathcal{F}} \eta(A)$. Let $\lim_{n \to \infty} \eta(A_n) = \alpha$. The key step of the proof is to find a set A^+ such that $\eta(A^+) = \alpha$. Although the sequence $\{\bigcup_{i=1}^{n} A_i\}_{n=1}^{\infty}$ is monotone increasing, $\eta\left(\bigcup_{i=1}^{n} A_i\right)$ is not necessarily as large as $\eta(A_n)$. However, the following trick replaces $\bigcup_{i=1}^{n} A_i$ by a sequence of sets whose signed measures do increase. For each n, partition Ω using the sets $A_1, \ldots, A_n$ and their complements. Let C_n be the union of all of the component sets that have positive signed measure. Since the $(n+1)$st partition is a refinement of the nth partition, we see that $C_{n+1} \cap C_n^C$ is a union of sets with positive signed measure and

$$\eta(A_n) \leq \eta(C_n) \leq \eta\left(C_n \bigcup C_{n+1}\right).$$

By induction, we then show that $A^+ = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} C_n$ has $\eta(A^+) = \alpha$. The conclusions now follow easily. ∎

Theorem 37 has an interesting consequence.

Lemma 38. *Suppose that μ and ν are finite and not mutually singular. Then there exists $\epsilon > 0$ and a set A with $\mu(A) > 0$ and $\epsilon\mu(E) \leq \nu(E)$ for every $E \subseteq A$.*

Proof: For each n, let $\eta_n = \nu - (1/n)\mu$. Let $\beta = \nu(\Omega)$. Let A_n^+ and be the set called A^+ in Theorem 37 when η is η_n. Let $M = \cap_{n=1}^{\infty} A_n^{+C}$. Since $\eta_n(E) \leq 0$ for every subset of A_n^{+C}, we have $\eta_n(M) \leq 0$ for all n and $\nu(M) \leq (1/n)\mu(M)$. It follows that $\nu(M) = 0$ and $\nu(M^C) = \beta$. Since μ and ν are not mutually singular, $\mu(M^C) > 0$ and at least one $\mu(A_n^+) > 0$. Let $A = A_n^+$ and $\epsilon = 1/n$. ■

Proof: [Theorem 30] The σ-finite case follows easily from the finite case, so assume that μ and ν are finite with $\nu \ll \mu$. Let $\mathcal{G}$ be the set of all nonnegative measurable functions g such that $\int_E g d\mu \leq \nu(E)$ for all $E \in \mathcal{F}$. Because $0 \in \mathcal{G}$, we know that $\mathcal{G}$ is nonempty. If g_1 and g_2 are in $\mathcal{G}$, we know that $\{g_1 \leq g_2\}$ is measurable, hence it is easy to see that $\max\{g_1, g_2\} \in \mathcal{G}$. Also, if $g_n \in \mathcal{G}$ for all n and $g_n \uparrow g$, then the monotone convergence theorem implies that $g \in \mathcal{G}$. So, let $\alpha = \sup_{g\in\mathcal{G}} \int g d\mu$ and let $\lim_{n\to\infty} \int g_n d\mu = \alpha$. Let $f_n = \max\{g_1, \ldots, g_n\}$ so that there is f such that $f_n \uparrow f$, $f_n \in \mathcal{G}$ for all n, and $\lim_{n\to\infty} \int f_n d\mu = \alpha$. It follows that $\int f d\mu = \alpha$ and $f \in \mathcal{G}$. Define $\nu_1(E) = \int_E f d\mu$ and $\nu_2 = \nu - \nu_1$, which is a measure since $\nu_1 \leq \nu$. If ν_2 and μ were not mutually singular, there would exist $\epsilon > 0$ and a set A with $\mu(A) > 0$ and $\epsilon\mu(E) \leq \nu_2(E)$ for all $E \subseteq A$. For each $E \in \mathcal{F}$,

$$\begin{aligned}\int_E (f + \epsilon I_A) d\mu &= \int_E f d\mu + \epsilon\mu(E \cap A) \\ &\leq \nu_1(E) + \nu_2(E \cap A) \leq \nu_1(E) + \nu_2(E) = \nu(E).\end{aligned}$$

Hence $h = f + \epsilon I_A \in \mathcal{G}$, but $\int h d\mu = \alpha + \epsilon\mu(A) > \alpha$, a contradiction. It follows that ν_2 and μ are mutually singular. Hence, there exists S such that $\nu_2(S) = \mu(S^C) = 0$. Since $\nu \ll \mu$, we have $\nu(S^C) = 0$. Because $\nu_2 \leq \nu$, we have $\nu_2(S^C) = 0$ and $\nu_2(\Omega) = 0$. It follows that $\nu = \nu_1$ and the proof of existence is complete. Uniqueness follows from part 3 of Theorem 21 in lecture notes set 3. ■