36-752 Advanced Probability Overview Spring 2018

4. Product Spaces

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Associated reading: Sec 2.6 and 2.7 of Ash and Doléans-Dade; Sec 1.7 and A.3 of Durrett.

1 Random Vectors and Product Spaces

We have already defined random variables and random quantities. A special case of the latter and generalization of the former is a random vector.

Definition 1 (Random Vector). *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X : \Omega \to I\!R^k$ be a measurable function. Then X is called a* random vector *.*

There arises, in this definition, the question of what σ-field of subsets of $\mathbb{R}^k$ should be used. When left unstated, we always assume that the σ-field of subsets of a multidimensional real space is the Borel σ-field, namely the smallest σ-field containing the open sets. However, because $\mathbb{R}^k$ is also a product set of k sets, each of which already has a natural σ-field associated with it, we might try to use a σ-field that corresponds to that product in some way.

1.1 Product Spaces

The set $\mathbb{R}^k$ has a topology in its own right, but it also happens to be a product set. Each of the factors in the product comes with its own σ-field. There is a way of constructing σ-field's of subsets of product sets directly without appealing to any additional structure that they might have.

Definition 2 (Product σ-Field). *Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces. Let $\mathcal{F}_1 \otimes \mathcal{F}_2$ be the smallest σ-field of subsets of $\Omega_1 \times \Omega_2$ containing all sets of the form $A_1 \times A_2$ where $A_i \in \mathcal{F}_i$ for $i = 1, 2$. Then $\mathcal{F}_1 \otimes \mathcal{F}_2$ is the* product σ-field.

Lemma 3. *Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces. Suppose that $\mathcal{C}_i$ generates $\mathcal{F}_i$ and $\Omega_i \in \mathcal{C}_i$ for $i = 1, 2$. Let $\mathcal{C} = \{C_1 \times C_2 : C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2\}$. Then $\sigma(\mathcal{C}) = \mathcal{F}_1 \otimes \mathcal{F}_2$. Furthermore, $\mathcal{C}$ is a π-system if both $\mathcal{C}_1$ and $\mathcal{C}_2$ are π-systems.*

Proof: Because $\sigma(\mathcal{C})$ is a σ-field, it contains all sets of the form $C_1 \times A_2$ where $A_2 \in \mathcal{F}_2$. For the same reason, it must contain all sets of the form $A_1 \times A_2$ for $A_i \in \mathcal{F}_i$ $(i = 1, 2)$. Because

$$(C_1 \times C_2) \cap (D_1 \times D_2) = (C_1 \cap D_1) \times (C_2 \times D_2),$$

we see that $\mathcal{C}$ is a π-system. ■

Example 4 (Product σ Field in Euclidean Space). *Let $\Omega_i = I\!R$ for $i = 1, 2$, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ both be $\mathcal{B}^1$. Let $\mathcal{C}_i$ be the collection of all intervals centered at rational numbers with rational lengths. Then $\mathcal{C}_i$ generates $\mathcal{F}_i$ for $i = 1, 2$ and the product topology is the smallest topology containing $\mathcal{C}$ as defined in Lemma 3. It follows that $\mathcal{F}_1 \otimes \mathcal{F}_2$ is the smallest σ-field containing the product topology. We call this σ-field $\mathcal{B}^2$.*

Example 5. *This time, let $\Omega_1 = I\!R^2$ and $\Omega_2 = I\!R$. The product set is $I\!R^3$ and the product σ-field is called $\mathcal{B}^3$. It is also the smallest σ-field containing all open sets in $I\!R^3$. The same idea extends to each finite-dimensional Euclidean space, with Borel σ-field's $\mathcal{B}^k$, for $k = 1, 2, \ldots$.*

The product σ-field is also the smallest σ-field such that the coordinate projection functions are measurable. The coordinate projection functions for a product set $S_1 \times S_2$ are the functions $f_i : S_1 \times S_2 \to S_i$ (for $i = 1, 2$) defined by $f_i(s_1, s_2) = s_i$ (for $i = 1, 2$).

Infinite-dimensional product spaces pose added complications that we will not consider until later in the course.

There are a number of facts about product spaces that we might take for granted.

Proposition 6 (Basic Properties of Product Spaces). *Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces.*

- *For each $B \in \mathcal{F}_1 \otimes \mathcal{F}_2$ and each $\omega_1 \in \Omega_1$, the ω_1-section of B, $B_{\omega_1} = \{\omega_2 : (\omega_1, \omega_2) \in B\}$ is in $\mathcal{F}_2$.*

- *If μ_2 is a σ-finite measure on $(\Omega_2, \mathcal{F}_2)$, then $\mu_2(B_{\omega_1})$ is a measurable function from Ω_1 to $I\!R$.*

- *If $f : \Omega_1 \times \Omega_2 \to S$ is measurable, then for every $\omega_1 \in \Omega_1$, the function $f_{\omega_1} : \Omega_2 \to S$ defined by $f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2)$ is measurable.*

- *If μ_2 is a σ-finite measure on $(\Omega_2, \mathcal{F}_2)$ and if $f : \Omega_1 \times \Omega_2 \to I\!R$ is nonnegative, then $\int f(\omega_1, \omega_2)\mu_2(d\omega_2)$ defines a measurable (possibly infinite valued) function of ω_1.*

To prove results like these, start with product sets or indicators of product sets and then show that the collection of sets that satisfy the results is a σ-field. Then, if necessary, proceed with the standard machinery. For example, consider the second claim. For the case of finite μ_2, the claim is true if B is a product set. It is easy to show that the collection $\mathcal{C}$ of all sets B for which $\mu_2(B_{\omega_1})$ is measurable is a λ-system. Then use π-λ Theorem. Here is the proof that the second claim holds for σ-finite measures once it is proven that it holds for finite measures. Let $\{A_n\}_{n=1}^{\infty}$ be disjoint elements of $\mathcal{F}_2$ that cover Ω_2 and have finite μ_2 measure. Define $\mathcal{F}_{2,n} = \{C \cap A_n : C \in \mathcal{F}_2\}$ and $\mu_{2,n}(C) = \mu_2(A_n \cap C)$ for all $C \in \mathcal{F}_2$. Then

$(A_n, \mathcal{F}_{2,n}, \mu_{2,n})$ is a finite measure space for each n and $\mu_{2,n}(B_{\omega_1})$ is measurable for all n and all B in the product σ-field. Finally, notice that

$$\mu_2(B_{\omega_1}) = \sum_{n=1}^{\infty} \mu_2(B_{\omega_1} \cap A_n) = \sum_{n=1}^{\infty} \mu_{2,n}(B_{\omega_1}),$$

a sum of nonnegative measurable functions, hence measurable. The standard machinery can be used to prove the third and fourth claims. (Even though the third claim does not involve integrals, the steps in the proof are similar to those of the standard machinery.)

Lemma 7. *Let $(\Omega_i, \mathcal{F}_i)$ and $(S_i, \mathcal{A}_i)$ be measurable spaces for $i = 1, 2$. Let $f_i : \Omega_i \to S_i$ be a function for $i = 1, 2$. Define $g(\omega_1, \omega_2) = (f_1(\omega_1), f_2(\omega_2))$, which is a function from $\Omega_1 \times \Omega_2$ to $S_1 \times S_2$. Then f_i is $\mathcal{F}_i/\mathcal{A}_i$-measurable for $i = 1, 2$ if and only if g is $\mathcal{F}_1 \otimes \mathcal{F}_2/\mathcal{A}_1 \otimes \mathcal{A}_2$-measurable.*

Proof: For the "only if" direction, assume that each f_i is measurable. It suffices to show that for each product set $A_1 \times A_2$ (with $A_i \in \mathcal{A}_i$ for $i = 1, 2$) $g^{-1}(A_1 \times A_2) \in \mathcal{F}_1 \otimes \mathcal{F}_2$. But, it is easy to see that $g^{-1}(A_1 \times A_2) = f_1^{-1}(A_1) \times f_2^{-1}(A_2) \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

For the "if" direction, suppose that g is measurable. Then for every $A_1 \in \mathcal{A}_1$, $g^{-1}(A_1 \times S_2) \in \mathcal{F}_1 \otimes \mathcal{F}_2$. But $g^{-1}(A_1 \times S_2) = f_1^{-1}(A_1) \times \Omega_2$. The fact that $f_1^{-1}(A_1) \in \mathcal{F}_1$ will now follow from the first claim in Proposition 6. So f_1 is measurable. Similarly, f_2 is measurable. ∎

Proposition 8. *Let $(\Omega, \mathcal{F})$, $(S_1, \mathcal{A}_1)$, and $(S_2, \mathcal{A}_2)$ be measurable spaces. Let $X_i : \Omega \to S_i$ for $i = 1, 2$. Define $X = (X_1, X_2)$ a function from Ω to $S_1 \times S_2$. Then X_i is $\mathcal{F}/\mathcal{A}_i$ measurable for $i = 1, 2$ if and only if X is $\mathcal{F}/\mathcal{A}_1 \otimes \mathcal{A}_2$ measurable.*

Lemma 7 and Proposition 8 extend to higher-dimensional products as well.

1.2 Product Measures

Product measures are measures on product spaces that arise from individual measures on the component spaces. Product measures are just like joint distributions of independent random variables, as we shall see after we define both concepts.

Theorem 9 (Product Measure). *Let $(\Omega_i, \mathcal{F}_i, \mu_i)$ for $i = 1, 2$ be σ-finite measure spaces. There exists a unique measure μ defined on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ that satisfies $\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ for all $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$.*

Proof: The uniqueness will follow from the uniqueness theorem since any two such measures will agree on the π-system of product sets. For the existence, consider the measurable function $\mu_2(B_{\omega_1})$ defined in Proposition 6. For $B \in \mathcal{F}_1 \otimes \mathcal{F}_2$, define

$$\mu(B) = \int \mu_2(B_{\omega_1}) d\mu_1.$$

It is straightforward to check that μ is a σ-finite measure. If B is a product set $A_1 \times A_2$, then $B_{\omega_1} = A_2$ for all ω_1, and

$$\mu(B) = \int \mu_2(A_2) I_{A_1}(\omega_1)\, \mu_1(d\omega_1) = \mu_1(A_1)\mu_2(A_2).$$

It follows that μ is the desired measure. ∎

Definition 10. *The measure μ in Theorem 9 is called the* product measure of μ_1 and μ_2 *and is sometimes denoted $\mu_1 \times \mu_2$.*

How to integrate with respect to a product measure is an interesting question. For nonnegative functions, there is a simple answer.

Theorem 11 (Fubini/Tonelli theorem). *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be σ-finite measure spaces. Let $f : \Omega_1 \times \Omega_2 \to I\!R$ be a nonnegative $\mathcal{F}_1 \otimes \mathcal{F}_2/\mathcal{B}^1$-measurable function. Then*

$$\int f d\mu_1 \times \mu_2 = \int \left[\int f(\omega_1, \omega_2)\mu_1(d\omega_1)\right] \mu_2(d\omega_2) = \int \left[\int f(\omega_1, \omega_2)\mu_2(d\omega_2)\right] \mu_1(d\omega_1). \quad (1)$$

Proof: We will use the standard machinery. If f is the indicator of a set B, then all three integrals in Equation (1) equal $\mu_1 \times \mu_2(B)$, as in the poof of Theorem 9. By linearity of integrals, the three integrals are the same for all nonnegative simple functions. Next, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative simple functions all $\leq f$ such that $\lim_{n\to\infty} f_n = f$. We have just shown that, for each n,

$$\int f_n d\mu_1 \times \mu_2 = \int \left[\int f_n(\omega_1, \omega_2)\mu_1(d\omega_1)\right] \mu_2(d\omega_2).$$

For each ω_2, the monotone convergence theorem says

$$\lim_{n\to\infty} \int f_n(\omega_1, \omega_2)\mu_1(d\omega_1) = \int f(\omega_1, \omega_2)\mu_1(d\omega_1).$$

Again, the monotone convergence theorem says that

$$\lim_{n\to\infty} \int \left[\int f_n(\omega_1, \omega_2)\mu_1(d\omega_1)\right] \mu_2(d\omega_2) = \int \left[\lim_{n\to\infty} \int f_n(\omega_1, \omega_2)\mu_1(d\omega_1)\right] \mu_2(d\omega_2).$$

Combining these last three equations proves that the first two integrals in Equation (1) are equal. A similar argument shows that the first and third are equal. ∎

Theorem 11 says that nonnegative product-measurable functions can be integrated in either order to get the integral with respect to product measure. A similar result holds for integrable product-measurable functions.

Corollary 12. *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be σ-finite measure spaces. Let $f : \Omega_1 \times \Omega_2 \to I\!R$ be a function that is integrable with respect to $\mu_1 \times \mu_2$. Then Equation* (1) *holds.*

The only sticky point in the proof of Corollary 12 is making sure that $\infty - \infty$ occurs with measure zero in the iterated integrals. But if $\infty(-\infty)$ occurs with positive measure for $f^+(f^-)$ in either of the iterated integrals, that iterated integral would be infinite and $f^+(f^-)$ would not be integrable.

Exercise 13. *Let X be a nonnegative random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ having distribution function F. Show that $\mathrm{E}(X) = \int_0^\infty [1 - F(x)]dx$.*

Example 14. *This example satisfies neither the conditions of Theorem 11 nor those of Corollary 12. Let*

$$f(x, y) = \begin{cases} x \exp(-[1 + x^2]y/2) & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \int f(x, y)dx &= \exp(-y/2) \int x \exp(-x^2 y/2) \\ &= 0, \\ \int f(x, y)dy &= x \int_0^\infty \exp(-[1 + x^2]y/2)dy \\ &= \frac{2x}{1 + x^2}. \end{aligned}$$

The iterated integral in one direction is 0 and is undefined in the other direction.

These results extend to arbitrary finite products.

Example 15. *The product of k copies of Lebesgue measure on $I\!R^1$ is Lebesgue measure on $I\!R^k$. Theorem 11 and Corollary 12 give sufficient conditions under which integrals can be performed in any desired order.*

2 Independence

We shall define what it means for collections of events and random quantities to be independent.

Definition 16 (Independence Between Collections of Sets). *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be subsets of $\mathcal{F}$. We say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are* independent *if $P(A_1 \cap A_2) = P(A_1)P(A_2)$ for all $A_1 \in \mathcal{C}_1$ and $A_2 \in \mathcal{C}_2$.*

Example 17. *If each of $\mathcal{C}_1$ and $\mathcal{C}_2$ contains only one event, then $\mathcal{C}_1$ being independent of $\mathcal{C}_2$ is the same as those events being independent.*

Definition 18 (Independence Between Random Variables/Quantities). *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $(S_i, \mathcal{A}_i)$ for $i = 1, 2$ be measurable spaces. Let $X_i : \Omega \to S_i$ be $\mathcal{F}/\mathcal{A}_i$ measurable for $i = 1, 2$. We say that X_1 and X_2 are* independent *if the σ-field's $\sigma(X_1)$ and $\sigma(X_2)$ (recall the σ-field generated by functions) are independent.*

Proposition 19. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are independent π-systems then $\sigma(\mathcal{C}_1)$ and $\sigma(\mathcal{C}_2)$ are independent.*

Example 20. *Let f_1 and f_2 be densities with respect to Lebesgue measure. Let P be defined on $(I\!R^2, \mathcal{B}^2)$ by $P(C) = \int\int_C f_1(x) f_2(y) dx dy$. Then the following two σ-field's are independent :*

$$\begin{aligned} \mathcal{C}_1 &= \{A \times I\!R : A \in \mathcal{B}^1\}, \\ \mathcal{C}_2 &= \{I\!R \times A : A \in \mathcal{B}^1\}. \end{aligned}$$

Also, the following two random variables are independent: $X_1(x, y) = x$ and $X_2(x, y) = y$ (i.e., the coordinate projection functions). Indeed, $\mathcal{C}_i = \sigma(X_i)$ for $i = 1, 2$.

Example 21. *Let X_1 and X_2 be two random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Suppose that the joint distribution of (X_1, X_2) has a density $f(x, y)$ that factors into $f(x, y) = f_1(x) f_2(y)$, the two marginal densities. Then, for each product set $A \times B$ with $A, B \in \mathcal{B}^1$,*

$$\begin{aligned} \Pr(X_1 \in A, X_2 \in B) &= \Pr((X_1, X_2) \in A \times B) \\ &= \int_A \int_B f_1(x) f_2(y) dy dx \\ &= \int_A f_1(x) dx \int_B f_2(y) dy \\ &= \Pr(X_1 \in A) \Pr(X_2 \in B). \end{aligned}$$

So, X_1 and X_2 are independent. The same reasoning would apply if the two random variables were discrete. It would also apply if one were discrete and the other continuous.

These definitions extend to more than two collections of events and more than two random variables.

Definition 22 (Independence of Multiple Collections of Subsets). *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{\mathcal{C}_t : t \in T\}$ be a collection of subsets of $\mathcal{F}$. We say that the $\mathcal{C}_t$'s are* (mutually) independent *if, for every finite integer $n \geq 2$ and no more than the cardinality of T, and for all distinct $t_1, \ldots, t_n \in T$, and $A_{t_i} \in \mathcal{C}_{t_i}$ for $i = 1, \ldots, n$,*

$$P\left(\bigcap_{i=1}^n A_{t_i}\right) = \prod_{i=1}^n P(A_{t_i}).$$

Definition 23 (Independence of Multiple Random Variables/Quantities). *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{(S_t, \mathcal{A}_t) : t \in T\}$ be measurable spaces. Let $X_t : \Omega \to S_t$ be $\mathcal{F}/\mathcal{A}_t$ measurable for each $t \in T$. We say that $\{X_t : t \in T\}$ are* (mutually) independent *if the σ-field's $\{\sigma(X_t) : t \in T\}$ are mutually independent.*

Theorem 24. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $(S_i, \mathcal{A}_i)$ for $i = 1, 2$ be measurable spaces. Let $X_1 : \Omega \to S_1$ and $X_2 : \Omega \to S_2$ be random quantities. Define $X = (X_1, X_2)$. The distribution of $X : \Omega \to S_1 \times S_2$, μ_X is the product measure $\mu_{X_1} \times \mu_{X_2}$ if and only if X_1 and X_2 are independent.*

Proof: For the "if" direction, suppose that X_1 and X_2 are independent. Then for every product set $A_1 \times A_2$,

$$\begin{aligned} \mu_X(A_1 \times A_2) &= \Pr(X_1 \in A_1, X_2 \in A_2) = \Pr(X_1 \in A_1) \Pr(X_2 \in A_2) \\ &= \mu_{X_1}(A_1)\mu_{X_2}(A_2). \end{aligned}$$

It follows from the uniqueness of product measure that μ_X is the product measure.

For the "only if" direction, suppose that $\mu_X = \mu_{X_1} \times \mu_{X_2}$. Then, for every $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$,

$$\begin{aligned} \Pr(X_1 \in A_1, X_2 \in A_2) &= \mu_X(A_1 \times A_2) = \mu_{X_1}(A_1)\mu_{X_2}(A_2) \\ &= \Pr(X_1 \in A_1) \Pr(X_2 \in A_2). \end{aligned}$$

■

3 Stochastic Process and Infinite Product Space

A stochastic process is an indexed collection of random quantities.

Definition 25 (Stochastic Processes). *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let T be a set. Suppose that, for each $t \in T$, there is a measurable space $(\mathcal{X}_t, \mathcal{F}_t)$ and a random quantity $X_t : \Omega \to \mathcal{X}_t$. The collection $\{X_t : t \in T\}$ is called a* stochastic process *, and T is called the* index set*.*

The most popular stochastic processes are those for which $\mathcal{X}_t = \mathbb{R}$ for all t. Among those, there are two very commonly used index sets, namely $T = \mathbb{Z}^+$ (sequences of random variables) and $T = \mathbb{R}^{+0}$ (continuous-time stochastic processes). There are, however, many more general index sets than these, and they are all handled in the same general fashion.

Example 26 (Random vector). *Let $T = \{1, \ldots, k\}$ and for each $i \in T$, let X_i be a random variable (all defined on the same probability space). Then $(X_1, \ldots, X_k)$ is one way to represent $\{X_i : i \in \{1, \ldots, k\}\}$.*

Example 27 (Random probability measure). *Let $\Theta : \Omega \to I\!R^k$ be a random vector with distribution μ_Θ. Let $f : I\!R \times I\!R^k \to I\!R^{+0}$ be a measurable function such that $\int f(x,\theta)dx = 1$ for all $\theta \in I\!R^k$. Let $T = \mathcal{B}^1$, the Borel σ-field of subsets of $I\!R$. For each $B \in T$, define*

$$X_B(\omega) = \int_B f(x, \Theta(\omega))dx.$$

The stochastic process $\{X_B : B \in \mathcal{B}^1\}$ is a random probability measure.

The distribution of a stochastic process is the probability measure induced on its range space. Unfortunately, if T is an infinite set, the range space of a stochastic process is an infinite-dimensional product set. We need to be able to construct a σ-field of subsets of such a set.

An infinite product of sets is usually defined as a set of functions.

Definition 28 (Alternative Representation of Product Set). *Let T be a set. Suppose that, for each $t \in T$, there is a set $\mathcal{X}_t$. The* product set *$\mathcal{X} = \prod_{t\in T} \mathcal{X}_t$ is defined to be the set of all functions $f : T \to \bigcup_{t\in T} \mathcal{X}_t$ such that, for every t, $f(t) \in \mathcal{X}_t$. When each $\mathcal{X}_t$ is the same set $\mathcal{Y}$, then the product set is denoted $\mathcal{Y}^T$.*

The above definition applies to all product sets, not just infinite ones.

Example 29. *It is easy to see that finite product sets can be considered sets of functions also. Each k-tuple is a function f from $\{1,\ldots,k\}$ to some space, where the ith coordinate is $f(i)$. For example, the notation $I\!R^k$ can be thought of as a shorthand for $I\!R^{\{1,\ldots,k\}}$. A vector $(x_1,\ldots,x_k)$ is the function f such that $f(i) = x_i$ for $i = 1,\ldots,k$.*

Example 30 (Random probability measure). *In Example 27, let $\mathcal{X}_B = [0,1]$ for all $B \in T$. Then each random variable X_B takes values in $\mathcal{X}_B$. The infinite product set is $[0,1]^{\mathcal{B}^1}$. Each probability measure on $(I\!R, \mathcal{B}^1)$ is a function from $\mathcal{B}^1$ into $[0,1]$. The product set contains other functions that are not probabilities. For example, the function $f(B) = 1$ for all $B \in \mathcal{B}^1$ is in the product set, but is not a probability.*

We want the σ-field of subsets of a product space to be large enough so that all of the coordinate projection functions are measurable.

Definition 31 (Projection, Cylinder Set, and Product Measure). *Let T be a set. For each $t \in T$, let $(\mathcal{X}_t, \mathcal{F}_t)$ be a measurable space. Let $\mathcal{X} = \prod_{t\in T} \mathcal{X}_t$ be the product set. For each $t \in T$, the t-*coordinate projection function *$\pi_t : \mathcal{X} \to \mathcal{X}_t$ is defined as $\pi_t(f) = f(t)$. A* one-dimensional cylinder set is a set of the form $\prod_{t\in T} B_t$ where there exists one $t_0 \in T$ and $B \in \mathcal{F}_{t_0}$ such that $B_{t_0} = B$ and $B_t = \mathcal{X}_t$ for all $t \neq t_0$. Define $\otimes_{t\in T}\mathcal{F}_t$ to be the σ-field generated by the one-dimensional cylinder sets, and call this the *product σ-field.*

Example 32. *Let $\mathcal{X} = \mathbb{R}^k$ for finite k. For $1 \le i \le k$, the i-coordinate projection function is $\pi_i(x_1, \ldots, x_k) = x_i$. An example of a one-dimensional cylinder set (in the case $k = 3$) is $\mathbb{R} \times [-3.7, 4.2) \times \mathbb{R}$.*

Example 33 (Random probability measure). *In Example 30, let Q be a probability on $\mathcal{B}^1$. Then Q is an element of the infinite product set $[0,1]^{\mathcal{B}^1}$. For each $B \in T$, the B-coordinate projection function evaluated at Q is $\pi_B(Q) = Q(B)$.*

Lemma 34. *The product σ-field is the smallest σ-field such that all π_t are measurable.*

Proof: Notice that, for each $t_0 \in T$ and each $B_{t_0} \in \mathcal{F}_{t_0}$, $\pi_{t_0}^{-1}(B_{t_0})$ is the one-dimensional cylinder set $\prod_{t \in T} B_t$ where $B_t = \mathcal{X}_t$ for all $t \neq t_0$. This makes every π_t measurable. Notice also that the sets required to make all the π_t measurable generate the product σ-field, hence the product σ-field is the smallest σ-field such that the π_t are all measurable. ∎

A stochastic process can be thought of as a random function. When a product space is explicitly considered a function space, the coordinate projection functions are sometimes called *evaluation functionals*.

Theorem 35. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let T be a set. For each $t \in T$, let $(\mathcal{X}_t, \mathcal{F}_t)$ be a measurable space and let $X_t : \Omega \to \mathcal{X}_t$ be a function. Let $\mathcal{X} = \prod_{t \in T} \mathcal{X}_t$. Define $\boldsymbol{X} : \Omega \to \mathcal{X}$ by setting $\boldsymbol{X}(\omega)$ to be the function f defined by $f(t) = X_t(\omega)$ for all t. Then $\boldsymbol{X}$ is $\mathcal{F}/\bigotimes_{t \in T} \mathcal{F}_t$-measurable if and only if each $X_t : \Omega \to \mathcal{X}_t$ is $\mathcal{F}/\mathcal{F}_t$-measurable.*

Proof: For the "if" direction, assume that each X_t is measurable. Let $\mathcal{C}$ be the collection of one-dimensional cylinder sets, which generates the product σ-field. Let $C \in \mathcal{C}$. Then there exists t_0 and $B \in \mathcal{F}_{t_0}$ such that $C = \prod_{t \in T} B_t$ where $B_{t_0} = B$ and $B_t = \mathcal{X}_t$ for all $t \neq t_0$. It follows that $\boldsymbol{X}^{-1}(C) = X_{t_0}^{-1}(B) \in \mathcal{F}$. So, $\boldsymbol{X}$ is measurable by Lemma 7 of Lecture Notes Set 2.

For the "only if" direction, assume that $\boldsymbol{X}$ is measurable. Let π_t be the t coordinate projection function for each $t \in T$. It is trivial to see that $X_t = \pi_t(\boldsymbol{X})$. Since each π_t is measurable, it follows that each X_t is measurable. ∎

The function $\boldsymbol{X}$ defined in Theorem 35 is an alternative way to represent the stochastic process $\{X_t : t \in T\}$. That is, instead of thinking of a stochastic process as an indexed set of random quantities, think of it as just another random quantity, but one whose range space is itself a function space. In this way, stochastic processes can be thought of as random functions. The idea is that, instead of thinking of X_t as a function of ω for each t, think of $\boldsymbol{X}(\omega)$ as a function of t for each ω.

Here are some examples of how to think of stochastic processes as random functions and vice-versa.

Example 36. *Let β_0 and β_1 be random variables. Let $T = I\!R$. For each $x \in I\!R$, define $X_x(\omega) = \beta_0(\omega) + \beta_1(\omega)x$. Define $\boldsymbol{X}$ as in Theorem 35. Then $\boldsymbol{X}$ is a random linear function. This means that, for every ω, $\boldsymbol{X}(\omega)$ is a linear function from $I\!R$ to $I\!R$. Indeed, it is the function that maps the number x to the number $\beta_0(\omega) + \beta_1(\omega)x$.*

Example 37 (Random probability measure). *In Example 27, define $\boldsymbol{X}(\omega)$ to be the function (element of the product set) that maps each set B to $\int_B f(x, \Theta(\omega))dx$. To see that $\boldsymbol{X} : \Omega \to [0,1]^{\mathcal{B}^1}$ is measurable, let C be the one-dimensional cylinder set $\prod_{B\in T} C_B$ where each $C_B = [0,1]$ except $C_{B_0} = D$. Define $g(\theta) = \int_{B_0} f(x,\theta)dx$. We know that $g : I\!R^k \to [0,1]$ is measurable. Hence $g(\Theta) : \Omega \to [0,1]$ is measurable. It follows that $\boldsymbol{X}^{-1}(C) = g^{-1}(D)$, a measurable set.*

Clearly, there must exist probability measures on product spaces such as $\left(\prod_{t\in T} \mathcal{X}_t, \otimes_{t\in T}\mathcal{F}_t\right)$. If we start with a stochastic process $\{X_t : t \in T\}$ and represent it as a random function $\boldsymbol{X}$, then the distribution of $\boldsymbol{X}$ is a probability measure on the product space. This distribution has the obvious marginal distributions for the individual X_t's. But, in general, nothing much can be said about other aspects of the joint distribution.

3.1 Kolmogorov's Extension

There is such a thing as product measure on an infinite product space, but to prove it, we need a little more machinery. There is a theorem that says that finite-dimensional distributions that satisfy a certain intuitive condition will determine a unique joint distribution on the product space. Here we will focus on probability measures.

Definition 38 (Finite Dimension Projection). *Let T be an index set. For each $t \in T$, there is a measurable space $(\mathcal{X}_t, \mathcal{F}_t)$. For all $v \subseteq T$, let $(\mathcal{X}_v, \mathcal{F}_v)$ be the corresponding product space and product σ-field. Let P_v be a probability measure on $(\mathcal{X}_v, \mathcal{F}_v)$. For $u \subset v \subseteq T$, the* projection *of P_v on $(\mathcal{X}_u, \mathcal{F}_u)$ is the probability measure $\pi_u(P_v)$ defined by*

$$[\pi_u(P_v)](B) = P_v(x \in \mathcal{X}_v : x_u \in B), \quad B \in \mathcal{F}_u.$$

Similarly, if Q is a probability measure on $\left(\prod_{t\in T} \mathcal{X}_t, \otimes_{t\in T}\mathcal{F}_t\right)$, the projection of Q on $(\mathcal{X}_v, \mathcal{F}_v)$ is defined by

$$[\pi_v(Q)](B) = Q\left[\omega \in \prod_{t\in T} \mathcal{X}_t : \omega_v \in B\right], \quad B \in \mathcal{F}_v.$$

Theorem 39 (Kolmogorov's Extension). *For each t in the arbitrary index set T, let $\mathcal{X}_t = I\!R$ and $\mathcal{F}_t$ the Borel sets of $I\!R$. Assume that for each finite nonempty set v of t, we are given a probability measure P_v on $\mathcal{F}_v$. Assume the P_v are* consistent, *that is,*

$$\pi_u(P_v) = P_u, \quad \text{for each nonempty } u \subset v.$$

Then there is a unique probability measure P on $\left(\prod_{t\in T} \mathcal{X}_t, \otimes_{t\in T}\mathcal{F}_t\right)$ such that $\pi_v(P) = P_v$ for all v.