36-752: Advanced Probability	**Spring 2018**

Lecture 2: February 1

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

Last time:

- let Ω be the universe set. $\mathcal{F}$ is a σ-field over Ω if it satisfies the following properties:
  1. $\Omega \in \mathcal{F}$,
  2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,
  3. $A_1, A_2, \ldots \Rightarrow \bigcup_n A_n \in \mathcal{F}$,

  then $(\Omega, \mathcal{F})$ is called measurable space;
- $\mathcal{F}$ is a field over Ω if it satisfies properties 1),2) listed above, and 3)$A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$

Example: field, but not σ-field.
Let $\mathcal{U}$ be the collection of union of disjoint intervals of the form $(b, +\infty)$, $\emptyset$, $(-\infty + \infty)$, $(a, b]$, where $-\infty \leq a < b$. $\mathcal{U}$ is a field, but not a σ-field. Indeed:

- $(a, b) = \bigcup_n (a, b - \frac{1}{n}] \notin \mathcal{U}$;
- $\{a\} = \bigcap_n [a - \frac{1}{n}, a] \notin \mathcal{U}$.

New material:

Definition 2.1 *Let $\mathcal{A}$ be a collection of subsets of Ω. We call $\sigma(\mathcal{A})$ the σ-field generated by $\mathcal{A}$.*

Definition 2.2 *The σ-field generated by all open sets A (or equivalently by all closed sets) of the form (a, b), $-\infty < a < b < +\infty$, is called Borel σ-field, and it is denoted $\mathcal{B}(\mathbb{R}) = \sigma(A)$.*

In $\mathbb{R}^k$ the Borel σ-field $\mathcal{B}^k$ is the σ-field generated by the open sets (union of the open balls), ie

$$\mathcal{B}(a, r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}.$$

More generally, if Ω is a topological space, the Borel σ-field is the σ-field generated by open sets.
However, there esist non-Borel sets.

Definition 2.3 *Let $(\Omega, \mathcal{F})$ be a measurable space. A measure on this space is a function $\mu : \mathcal{F} \to \bar{\mathbb{R}}_+$ satisfying*

1. $\mu(\emptyset) = 0$;
2. *$A_1, A_2, \ldots$ are disjoint sets in $\mathcal{F} \Rightarrow \mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n)$ (countable additivity).*

- if $\mu(\Omega) < +\infty$ the measure is finite, infinite otherwise;
- the triple $(\Omega, \mathcal{F}, \mu)$, where Ω is the universe set, $\mathcal{F}$ is a σ-field on Ω and μ is a measure on the $(\Omega, \mathcal{F})$ is called measure space;
- if $\mu(\Omega) = 1$, μ is a probability.

If $\mathcal{F}$ is a field, then a measure on $\mathcal{F}$ is a measure on a field.

Example: Let $\mathcal{F} = 2^{\Omega}$ and define $\forall A \subset \mathcal{F}, \mu(A) = |A|$.

Definition 2.4 *A measure space is σ-finite if there exists a sequence of measurable sets $A_1, A_2, \ldots$ such that $\mu(A_n) < \infty$ $\forall n \geq 1$ and $\bigcup_n A_n = \Omega$.*

Example: non σ-finite measure.

- counting measure on uncountable set;
- a measure s.t. $\mu(\emptyset) = 0, \mu(A) = \infty \forall A \neq \emptyset$.

Basic properties of measures: Let $(\Sigma, \mathcal{F}, \mu)$ be a measure space, then

1. $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.
   Proof: $B = A \cup (B \setminus A)$, hence $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$.
2. $\mu(A) < \infty$ and $\mu(B) < \infty \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.
3. (countable sub-additivity:) $A_1, A_2, \ldots$ countable sequence of measurable sets $\Rightarrow \mu\left(\bigcup_n A_n\right) \leq \sum_n \mu(A_n)$.

   Proof: let $B_1 = A_1$ and for $n \geq 2$ let $B_n = A_n \cap A_{n-1}^c \cap A_{n_2}^c \cap \cdots \cap A_1^c = A_n \setminus \bigcup_{i=1}^{n-1} B_i$.

   We have $\bigcup_n A_n = \bigcup_n B_n$, and $A_n = B_n \cup (A_n \cap \bigcup_{i=1}^{n-1} B_i)$.
   Hence
   $$\mu(\bigcup_n A_n) = \mu(\bigcup_n B_n) = \sum_n \mu(B_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) \leq \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) = \sum_n (A_n).$$
4. (monotonicity and continuity of μ:) $A_1, A_2, \ldots$ monotone sequence of measurable sets $\Rightarrow \mu(\lim_n A_n) = \lim_n \mu(A_n)$ if

- A_n is increasing or
- A_n is decreasing and $\exists\, i \in \mathbb{N}$ s.t. $\mu(A_i) < \infty$.

Proof:

- let $A_\infty = \bigcup_n A_n = \lim_n A_n$. Let $A_1 = B_1$ and, for $n \geq 2$, $B_n = A_n \setminus A_{n-1}$. Moreover $A_n = \bigcup_{i=1}^n B_i$ and $\bigcup_n A_n = \bigcup_n B_n$. Hence

$$\mu(A_\infty) = \mu\left(\lim_n A_n\right) = \mu(\bigcup_n A_n) = \sum_n \mu(B_n) = \lim_n \sum_{i=1}^n \mu(B_i) = \lim_n \mu(A_n).$$

- w.l.o.g. assume that $\mu(A_1) < \infty$. Then, since $A_n \downarrow \bigcap_n A_n = A_\infty$, $A_1 \setminus A_n \uparrow A_1 \setminus A_\infty$ and by the finiteness of $\mu(A_1)$ we obtain $\mu(A_1 \setminus A_n) \uparrow \mu(A_1 \setminus A_\infty)$ as $n \to \infty$. Hence

$$\mu(A_1) - \mu(A_n) \uparrow \mu(A_1) - \mu(A_\infty) \iff \mu(A_n) \downarrow \mu(A_\infty).$$

Definition 2.5 *A property over the elements of Ω is said to hold almost surely (a.s.) if it holds over a measurable set A such that $\mu(A^c) = 0$.*

Equivalently, the measure is said to hold almost everywhere (a.e.).

Lemma 2.6 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $A_1, A_2, \ldots$ be a sequence of measurable sets. Assume μ to be finite. Then*

1. $\mu(\liminf_n A_n) \leq \liminf_n \mu(A_n) \leq \limsup_n \mu(A_n) \leq \mu(\limsup_n (A_n)$*;*
2. *if* $\lim_n A_n = A$*, then* $\lim_n \mu(A_n) = \mu(A)$.

Proof: 2) follows from 1).
1) Let $B_n = \bigcap_{i=n}^\infty A_i$ and $C_n = \bigcup_{i=n}^\infty A_i$. Then $B_n \uparrow \liminf_n A_n$ and $C_n \downarrow \limsup_n A_n$. Hence

$$\mu(A_n) \geq \mu(B_n)\ \forall n \Rightarrow \liminf_n \mu(A_n) \geq \liminf_n \mu(B_n) = \lim_n \mu(B_n) = \mu(\liminf_n A_n).$$

Similarly,

$$\mu(A_n) \geq \mu(B_n) \forall n \Rightarrow \limsup_n \mu(A_n) \geq \limsup_n \mu(C_n) = \lim_n \mu(C_n) = \mu(\limsup_n A_n).$$

Uniqueness

Example: Let $(\mathbb{R}, \mathcal{B})$ be a measurable space, and define on it the measure $\mu((-\infty, a]) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a \mathrm{e}^{-\frac{x^2}{2}} dx$. Does there exist another probability distribution that agrees on all the sets of the form $(-\infty, a]$, $a \in \mathbb{R}$? The answer is no.

Definition 2.7 *Let Ω be the universe set. A collection $\mathcal{A}$ of subsets of Ω is called:*

- *π-system if* $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$;

- λ-*system if*
  - $\Omega \in \mathcal{A}$,
  - $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,
  - *if* $A_1, A_2, \ldots$ *is a sequence of disjoint sets in* $\mathcal{A}$*, then* $\bigcup_n A_n \in \mathcal{A}$.

Example: $\mathcal{A} = \{(-\infty, a], a \in \mathbb{R}\}$ is a π-system.

Theorem 2.8 *(Uniqueness) If* μ_1 *and* μ_2 *are measures on a measurable space* $(\Omega, \mathcal{F})$ *s.t.*

1. $\mathcal{F} = \sigma(\mathcal{F})$ *is some* Π-*system,*
2. μ_1 *and* μ_2 *are* σ-*finite and they agree on* Π

then they agree on $\mathcal{F}$*.*