STAT 36-752: Advanced Probability Overview **Spring 2018**

Lecture 1: February 13

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

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This lecture's notes illustrate some uses of various LaTeX macros. Take a look at this and imitate.

1.1 Last Time

$f : \Omega \to \mathbb{R}$ with measure space $(\Omega, \mathcal{F}, \mu)$

We want to define the notion of an integral "$\int_\Omega f(w) d\mu(\omega)$"

1.2 Simple Functions

$f(\omega) = \sum_{i=1}^n a_i \mathbb{I}_{A_i}(w)$ where $\{a_1, a_2, \dots a_n\}$ are distinct reals and $\{A_1, A_2, \dots A_n\}$ is a partition of Ω.

Aside: Motivation on Randomness in measure theoretic probability: *"We are concerned with defining properties of probability that are coherent and consistent. We will learn the grammar of probability in this class."*

Definition 1.1 *The integral of a simple function in canonical form is:*

$$\int f d\mu = \int_\Omega f(w) d\mu(\omega) \tag{1.1}$$

$$= \sum_{i=1}^n a_i \mathbb{I}_{A_i}(w) \tag{1.2}$$

Where values of $\{-\infty, \infty\}$ *are allowed*

We note that by this definition, we can have 3 possible outcomes:

1. If $\int f d\mu < +\infty$ then it *exists*
2. If $\int f d\mu \in \{-\infty, \infty\}$ then it *does not exist.* In this case we say that f is *not integrable.*
3. $\int f d\mu$ is *undefined* otherwise

1.2.1 Conventions

We adopt the following conventions for our calculations

1. $+\infty = \infty$
2. $\infty \times 0 = 0$
3. $\infty + \infty = \infty$
4. $-\infty - \infty = -\infty$
5. $x \times \infty = \text{sign}(x) \times \infty, \forall x \in \mathbb{R}$
6. $\infty - \infty$ is undefined

Definition 1.2 (Integral of a non-negative measurable function)

$$\int f d\mu = \sup_{\substack{\phi \text{ simple} \\ 0 \le \phi \le f}} \int \phi d\mu \tag{1.3}$$

$$= \sup_{\substack{A_1, A_2, \dots A_n \\ \text{finite partition of } \Omega}} \sum_{i=1}^{n} \mu(A_i) \times (\inf f(\omega)) \tag{1.4}$$

Definition 1.3

$$\int f d\mu = \int f^+ d\mu + \int f^- d\mu \tag{1.5}$$

$$f^+(\omega) = \max\{0, f(\omega)\} \tag{1.6}$$

$$f^-(\omega) = -\min\{0, f(\omega)\} \tag{1.7}$$

We note the following based on the above definition:

- f is integrable if **both** f^+ and f^- are integrable
- If either f^+ or f^- have infinite integral then f has infinite integral
- f is integrable if when $\int |f| d\mu$ is integrable since $|f| = f^+ + f^-$

Lemma 1.4 *If $f \le g$ a.e $[\mu]$ then $\int f \le \int g$*

Note that we say a property holds a.e $[\mu]$ if $\exists$ a measurable set $A \subseteq \Omega$ s.t $\mu(A^c) = 0$ and the property does not hold on A^c.

Proof: Assume $f \ge 0, g \ge 0$. Let $A = \{\omega \in \Omega \mid f(\omega) \le g(\omega)\} \implies \mu(A^c) = 0$. Let $\{A_1, A_2, \dots A_n\}$ be a partition of Ω. Now we have:

$$\sum_{i=1}^{n} \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i) = \sum_{i=1}^{n} \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i \cap A) \tag{1.8}$$

$$\le \sum_{i=1}^{n} \left[\inf_{\omega \in A_i \cap A} f(\omega)\right] \mu(A_i \cap A) \qquad \text{(taking inf over smaller set } A_i \cap A)$$

$$= \sum_{i=1}^{n} \left[\inf_{\omega \in A_i \cap A} g(\omega)\right] \mu(A_i \cap A) \qquad \text{(since } f \le g \text{ a.e } [\mu])$$

$$\le \int g d\mu \qquad \text{(Think about } \{A_1 \cap A, A_2 \cap A, \dots A_n \cap A\} \text{ and } A^c)$$

$$\Rightarrow \int f d\mu \le \int g d\mu \tag{1.9}$$

■

Corollary 1.5 *If f and g are integrable then*

$$|\int (f-g)\, d\mu| \leq \int |f-g| d\mu$$

Proof: Homework Exercise! ■

Integrals can express sums. Let μ be a counting measure on Ω. If $A \subseteq \Omega \quad A$ measurable. Then $f = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ is the canonical form. Then $\int f d\mu = \sum_{\omega} f(\omega)$. If μ is the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$ then $\int f d\mu$ is the Lebesgue integral.

1.2.2 Riemann vs. Lebesgue Integral

$$\int_A f d\mu = \int_\Omega \mathbb{I}_A(\omega) f(\omega) d\mu$$

The Lebesgue integral is defined over a broader class of sets.

Theorem 1.6 *If f is continuous on $[a,b]$ and has a Riemann integral then it is equal to the Lebesgue integral. If f is bounded it is Riemann integrable if and only if the set of discontinuities of f has 0 Lebesgue measure and the 2 integrals coincide*

Claim If $f : I = [a, \infty) \to \mathbb{R}$ is Lebesgue integrable over $[a,b] \quad \forall b \geq a$ and $\int_a^b |f| d\mu \leq M$ for some $M > 0, b \geq a$ then f is Lebesgue integrable over I and

$$\lim_{b\to\infty} \int_a^b f(x) d(x) = \int_I f(x) d(x)$$

Example $f(x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}, a \leq b$. We then have

$$\int_a^b f(x) d(x) = \arctan b - \arctan a \leq \pi \tag{1.10}$$

$$\int_{-\infty}^{+\infty} f(x) d(x) = \lim_{a\to-\infty} \int_a^0 f(x) d(x) + \lim_{b\to\infty} \int_0^b f(x) d(x) \tag{1.11}$$

$$= \pi \tag{1.12}$$

Problem It may happen that $f(x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}, a \leq b$. We then have $\int_a^b f(x) d(x)$ exists and equals the Riemann integral and $\lim_{b\to\infty} \int_a^b f(x) d(x)$ also exists. This means that f has an **improper** Riemann integral but $\int_a^b |f| d\mu$ may not exist!

Example $I = [0, \infty) \quad f(x) = \frac{(-1)^n}{n}, \quad n-1 \le x < n$. If $b > 0$, let $m = \lceil b \rceil$ least integer $\ge b$. We then have

$$\int_a^b f(x)d(x) = \int_0^m f + \int_m^b f \tag{1.13}$$

$$= \sum_{i=1}^{n} \frac{(-1)^n}{n} + \frac{b-m}{m+1}(-1)^n \tag{1.14}$$

$$\to \log 2 \text{ as } b \to \infty \tag{1.15}$$

$$\text{But } \int_0^m |f|dx = \infty \to \text{ not the Lebesgue integral!} \tag{1.16}$$

1.2.3 Properties of Integrals

1. If $f \ge 0$ a.e. $[u]$ then $\int f d\mu \ge 0$
2. If $f = g$ a.e. $[u]$ and $\int f d\mu$ or $\int g d\mu$ exists so does the other and they equal each other
3. What about $f + g$? If f, g are both integrable then $\int (f+g)d\mu = \int f d\mu + \int g d\mu$

1.2.4 Limit Theorems and Standard Machinery

1. **Fatou's Lemma**
2. **Monotone Convergence Theorem**
3. **Dominated Convergence Theorem**

Lemma 1.7 (Fatou's) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions. Then we have*

$$\int_\Omega \liminf_n f_n(\omega) \le \liminf_n \int_\Omega f_n(\omega) d\omega$$

Theorem 1.8 (Monotone Convergence Theorem) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions. Let f be a measurable function such that:*

1. $f_n \le f \quad \forall n$ *a.e.* $[u]$
2. $\lim_{n\to\infty} f_n = f$ *a.e.* $[u]$

Then $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$

Proof:

$$f_n \le f \quad \forall n \text{ a.e. } [u] \tag{1.17}$$

$$\Rightarrow \int f_n d\mu \le \int f d\mu \quad \forall n \tag{1.18}$$

$$\liminf_n \int f_n d\mu \le \limsup_n \underbrace{\int f_n d\mu}_{\le \int f d\mu} \tag{1.19}$$

$$\le \int f d\mu \tag{1.20}$$

Also

$$\int \underbrace{\liminf_n f_n d\mu}_{=f} \leq \liminf_n \int f d\mu \qquad \text{(By Fatou's Lemma)}$$

$$= \int f d\mu \tag{1.21}$$

$$\Rightarrow \limsup_n \int f_n d\mu \leq \int f d\mu \tag{1.22}$$

$$\leq \liminf_n \int f_n d\mu \tag{1.23}$$

$$\tag{1.24}$$

Combining the above we have:

$$\int f d\mu = \liminf_n \int f_n d\mu \tag{1.25}$$

$$= \limsup_n \int f_n d\mu \tag{1.26}$$

$$= \lim_n \int f_n d\mu \tag{1.27}$$

$$\tag{1.28}$$

■

1.2.5 Application : Standard Machinery

Theorem 1.9 *If* $\int f d\mu$ *and* $\int g d\mu$ *exist then* $\int (f+g) d\mu = \int f d\mu + \int g d\mu$

The standard machinery is used to demonstrate the integrability of a class of functions as follows:

1. Prove the result for non-negative simple functions in the class
2. Prove it for non-negative measurable functions in the class, using the fact that if $f \geq 0 \quad \exists$ simple functions $f_n \geq 0$ s.t. $f_n \uparrow f$ and apply Monotone Convergence Theorem.
3. Do this for positive and negative part of the functions in the class