36-752: Advanced Probability Overview **Spring 2018**

Lecture 6: February 15

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6.1 Basic Limit Theorems

There are three limit theorems that are often used:

Theorem 6.1 (Fatou's Lemma) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions. Then*

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu.$$

Theorem 6.2 (Monotone Convergence Theorem) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions, and let f be a measurable function such that $f_n \leq f$ and $\lim_{n\to\infty} f_n = f$, a.e. $[\mu]$. Then*

$$\liminf_{n\to\infty} \int f_n d\mu \leq \limsup_{n\to\infty} \int f_n d\mu \leq \int f d\mu.$$

Theorem 6.3 (Dominated Convergence Theorem) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions, and let f and g be measurable functions such that $f_n \to f$ a.e. $[\mu]$, $|f_n| \leq g$ a.e. $[\mu]$, and $\int g d\mu < \infty$. Then,*

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Remarks: The monotone convergence theorem is slightly different from the version in many textbooks. They are actually similar and the version we present here is more general. The proofs can be found in notes.

6.2 The Standard Machinery

Standard machinery is a way of demonstrating integrability problems. We will use the proof for linearity of integrals to demonstrate the usage of it.

Lemma 6.4 (Linearity of Integrals) *If $\int f d\mu$ and $\int g d\mu$ exist, then $\int (f+g) d\mu = \int f d\mu + \int g d\mu$.*

Proof: We prove the claim in three steps

Step 1: Assume f and g are nonnegative simple functions, and show the claim.

Step 2: Assume f and g are non-negative measurable functions. Then there exist nonnegative simple functions $\{f_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ such that $f_n \uparrow f$ and $g_n \uparrow g$. Then we have $(f_n + g_n) \uparrow (f + g)$ and the claim follows from monotone convergence theorem.

Step 3: Assume f and g are measurable functions. Then we have $f = f^+ - f^-$ and $g = g^+ - g^-$ and $\int f^+ d\mu$, $\int f^- d\mu$, $\int g^+ d\mu$ and $\int g^- d\mu$ are finite. Then

$$\begin{aligned}\int (f+g)^+ d\mu + \int f^- d\mu + \int g^- d\mu &= \int \left[(f+g)^+ + f^- + g^-\right] d\mu \\ &= \int \left[(f+g)^- + f^+ + g^+\right] d\mu \\ &= \int (f+g)^- d\mu + \int f^+ d\mu + \int g d\mu,\end{aligned}$$

and the claim follows for f and g. ∎

Change of variable formula is another example for using standard machinery to prove measure theory properties. We present the formula here and proof can be found in the notes(using standard machinery).

Lemma 6.5 (Change of Variable) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $(\mathcal{S}, \mathcal{A})$ be a measurable apace. Let $f : \Omega \to \mathcal{S}$ be a measurable function. Let ν be a measure on image space $(\mathcal{S}, \mathcal{A})$ induced by μ and f, i.e., $\nu(A) = \mu(f^{-1}(A)), \forall A \in \mathcal{A}$. Let $g : \mathcal{S} \to \mathbb{R}$ be a measurable function. Then*

$$\int g d\nu = \int f(f) d\mu = \int g(f(\omega)) d\mu(\omega).$$

We can get a widely-used corollary from Lemma 6.5.

Corollary 6.6 (Law of Unconscious Statisticians) *If $X : \Omega \to \mathbb{R}$ is a random variable with distribution μ_X, and $f : \mathbb{R} \to \mathbb{R}$. Then*

$$\mathbb{E}\left[f(X)\right] = \int f(x) d\mu_X.$$

6.3 Additional Properties of Integrals

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f, g be extended real value functions. Then we have,

1. If $f \geq 0$ and $\mu\left(\{\omega : f(\omega) > 0\}\right) > 0$, then $\int f d\mu > 0$;
2. If f, g are integrable, and $\int_A f = \int_A g$, $\forall A \in \mathcal{A}$,