36-752: Advanced Probability **Spring 2018**

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7.1 Measure on Product Spaces

7.1.1 Measurable product spaces

Definition 7.1 (Product σ-Field) *Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. The **product σ-field** $\mathcal{F}_1 \otimes \mathcal{F}_2$ on $\Omega_1 \times \Omega_2$ is defined as the σ-field generated by the collection of all sets of the form $\{A_1 \times A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$. The sets in this collection are called **measurable rectangles**.*

Remark 7.2 *$\mathcal{F}_1 \otimes \mathcal{F}_2 \neq \mathcal{F}_1 \times \mathcal{F}_2$ because $\mathcal{F}_1 \times \mathcal{F}_2$ may not be closed on A^c or $A_1 \cup A_2$. (Consider $(\mathbb{R}^2, \mathcal{B}^2)$.)*

Remark 7.3 *The collection of measurable rectangles is a π-system.*

Definition 7.4 (Coordinate Projection) *For $i = 1, 2$ the **coordinate projection** $\pi_i : \Omega_1 \times \Omega_2 \mapsto \Omega_i$ is defined as $\pi_i(\omega_1, \omega_2) = \omega_i$.*

Claim 7.5 *$\mathcal{F}_1 \otimes \mathcal{F}_2$ is the smallest σ-field such that the coordinate projections are all measurable.*

Claim 7.6 *The k dimensional Borel σ-field satisfies $\mathcal{B}^k = \mathcal{B}^1 \otimes ... \otimes \mathcal{B}^1$.*

Proposition 7.7 (Properties) *Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces:*

- *For each $B \in \mathcal{F}_1 \otimes \mathcal{F}_2$ and each $\omega_1 \in \Omega_1$ the ω_1-section of B, $B_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in B\}$ is in $\mathcal{F}_2$.*
- *If μ_2 is a σ-field on $(\Omega_2, \mathcal{F}_2)$ then $\forall B \in \mathcal{F}_1 \otimes \mathcal{F}_2$ the function $f : \Omega_1 \mapsto \mathbb{R}$ defined by $f(\omega_1) = \mu_2(B_{\omega_1})$ is measurable.*
- *If $f : \Omega_1 \times \Omega_2 \mapsto (S, \mathcal{A})$ is measurable then $\forall \omega_1 \in \Omega_1$ the function $f_{\omega_1} : \Omega_2 \mapsto S$ defined by $f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2)$ is measurable.*
- *If μ_2 is σ-finite on $(\Omega_2, \mathcal{F}_2)$ and $f : \Omega_1 \times \Omega_2 \mapsto \mathbb{R}$ be measurable and nonnegative then the function $g : \Omega_1 \mapsto \bar{\mathbb{R}}^{0+}$ defined by $g(\omega_1) = \int f(\omega_1, \omega_2) d\mu_2(\omega_2)$ is measurable.*

Proof of the first property:

Proof: Fix $\omega_1 \in \Omega_1$. Let $C_{\omega_1} = \{B \in \mathcal{F}_1 \otimes \mathcal{F}_2 : B_{\omega_1} \in \mathcal{F}_2\}$. First show that C_{ω_1} is a σ-field :

- For $B = \Omega_1 \times \Omega_2$, $B_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2\} = \Omega_2 \in \mathcal{F}_2$.
- For $B \in C_{\omega_1}$, we have $B_{\omega_1} \in \mathcal{F}_2$ thus $B^c_{\omega_1} \in \mathcal{F}_2$. Consider $(B^c)_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in B^c\}$. Recall that $B_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in B\}$. We have $(B^c)_{\omega_1} \cap B_{\omega_1} = \emptyset$ and $(B^c)_{\omega_1} \cup B_{\omega_1} = \Omega_2$. Hence $(B^c)_{\omega_1} = B^c_{\omega_1}$ and $(B^c)_{\omega_1} \in \mathcal{F}_2$. Since $B \in C_{\omega_1} \subset \mathcal{F}_1 \otimes \mathcal{F}_2$ we have $B^c \subset \mathcal{F}_1 \otimes \mathcal{F}_2$. Therefore, $B^c \in C_{\omega_1}$.
- Consider $B = \bigcup_{n=1}^{\infty} B_n$ where $B_n \in C_{\omega_1}$ for all n, i.e. $(B_n)_{\omega_1} \in \mathcal{F}_2$ for all n. We have $B_n \in \mathcal{F}_1 \otimes \mathcal{F}_2$ for all n so $B \in \mathcal{F}_1 \otimes \mathcal{F}_2$. We will show $B_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in \bigcup_{n=1}^{\infty} B_n\} = \bigcup_{n=1}^{\infty} (B_n)_{\omega_1}$:

  For any $\omega_2 \in B_{\omega_1}$ there exists n such that $(\omega_1, \omega_2) \in B_n$. Hence $\omega_2 \in (B_n)_{\omega_1}$ and $\omega_2 \in \bigcup_{n=1}^{\infty} (B_n)_{\omega_1}$.

  For any $\omega_2 \in \bigcup_{n=1}^{\infty} (B_n)_{\omega_1}$ there exists n such that $\omega_2 \in (B_n)_{\omega_1}$. Hence $(\omega_1, \omega_2) \in B_n$ and $(\omega_1, \omega_2) \in \bigcup_{n=1}^{\infty} B_n$, which means $\omega_2 \in B_{\omega_1}$.

  Now we have shown that $B_{\omega_1} = \bigcup_{n=1}^{\infty} (B_n)_{\omega_1} \in \mathcal{F}_2$ which indicates that $B \in C_{\omega_1}$ holds.

Therefore C_{ω_1} is a σ-field .

Now consider $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2$. Then we have $(A_1 \times A_2)_{\omega_1} = A_2$ if $\omega_1 \in A_1$ and $(A_1 \times A_2)_{\omega_1} = \emptyset$ if $\omega_1 \notin A_1$. In any case we have $(A_1 \times A_2)_{\omega_1} \in \mathcal{F}_2$ and thus $A_1 \times A_2 \in C_{\omega_1}$. Therefore, all measurable rectangles in the form of $A_1 \times A_2$ are contained in C_{ω_1}. According to the fact that C_{ω_1} is a σ-field and the definition of $\mathcal{F}_1 \otimes \mathcal{F}_2$ we have $C_{\omega_1} \supset \mathcal{F}_1 \otimes \mathcal{F}_2$. We also have $C_{\omega_1} \subset \mathcal{F}_1 \otimes \mathcal{F}_2$ by definition. Therefore $C_{\omega_1} = \mathcal{F}_1 \otimes \mathcal{F}_2$, which means, for all $B \in \mathcal{F}_1 \otimes \mathcal{F}_2$, $B_{\omega_1} \in \mathcal{F}_2$ holds.

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Lemma 7.8 *Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$, $(S_1, \mathcal{A}_1)$ and $(S_2, \mathcal{A}_2)$ be measurable spaces. For $i = 1, 2$ let $f_i : \Omega_i \mapsto S_i$ be a function. Define function $g : \Omega_1 \times \Omega_2 \mapsto S_1 \times S_2$ by $g(w_1, w_2) = (f_1(\omega_1), f_2(\omega_2))$. Then g is $\mathcal{F}_1 \otimes \mathcal{F}_2 / \mathcal{A}_1 \otimes \mathcal{A}_2$ measurable if and only if f_i is $\mathcal{F}_i / \mathcal{A}_i$ measurable for $i = 1, 2$.*

7.1.2 Product measures

Theorem 7.9 (Product measure) *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measurable spaces where μ_1 and μ_2 are σ-finite measures. There exists a unique measure μ on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ that satisfies $\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ for all $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$. This measure is called the* ***product measure****, written as $\mu = \mu_1 \times \mu_2$.*

Proof:

Uniqueness:

First we show that any such measure must be σ-finite. Since μ_1 and μ_2 are σ-finite there exist $\{A_n\}_{n=1}^{\infty} \in \mathcal{F}_1$ and $\{B_n\}_{n=1}^{\infty} \in F_2$ such that $\bigcup_{n=1}^{\infty} A_n = \Omega_1$, $\bigcup_{n=1}^{\infty} B_n = \Omega_2$, $\mu_1(A_n)$ and $\mu_2(B_n)$ are finite for all n. Consider $\bigcup_{(i,j) \in \mathbb{N}^2} A_i \times B_j$. For and $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ there exists i, j such that $\omega_1 \in A_i$ and $\omega_2 \in B_j$, which means $(\omega_1, \omega_2) \in A_i \times b_j$. Hence $\bigcup_{(i,j) \in \mathbb{N}^2} A_i \times B_j = \Omega_1 \times \Omega_2$. For any $(i, j) \in \mathbb{N}^2$ we have $\mu(A_i \times B_j) = \mu_1(A_i)\mu_2(B_j) < \infty$. Since $\mathbb{N}^2$ is a countable set we can conclude that μ is σ-finite.

Suppose there are two measures μ and μ' satisfying the condition in the theorem. Recall that the collection of measurable rectangles $\{A_1 \times A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$ is a π-system. μ and μ' are both σ-finite and agree on this π-system. By Uniqueness theorem they agree on the generated σ-field $\mathcal{F}_1 \otimes \mathcal{F}_2$, i.e., $\mu = \mu'$, which means such measure must be unique.

Existence:

For any $B \in \mathcal{F}_1 \otimes \mathcal{F}_2$ let $\mu(B) = \int_{\Omega_1} \mu_2(B_{\omega_1}) d\mu_1(\omega_1)$ where $B_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2\}$ as introduced previously. Then μ is a measure.

For any $A_1 \in \mathcal{F}_1$, $A_2 \in \mathcal{F}_2$,

$$\mu(A_1 \times A_2) = \int_{\Omega_1} \mu_2((A_1 \times A_2)_{\omega_1}) d\mu_1(\omega_1) = \int_{\Omega_1} \mathbb{1}_{A_1} \mu_2(A_2) d\mu_1(\omega_1) = \mu_2(A_2) \int_{\Omega_1} \mathbb{1}_{A_1} d\mu_1(\omega_1) = \mu_1(A_1)\mu_2(A_2)\,.$$

Hence such measure exists.

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Theorem 7.10 (Tonelli/Fubini theorem) *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measurable spaces where μ_1 and μ_2 are σ-finite measures. Let $\mu = \mu_1 \times \mu_2$ be the product measure on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. Let $f : \Omega_1 \times \Omega_2 \mapsto \mathbb{R}$ be a nonnegative measurable function. (Can be extended to integrable functions with respect to the product measure μ, i.e. $\int |f| d\mu < \infty$.) Then the following holds:*

$$\int f d\mu = \int \left[\int f(\omega_1, \omega_2) d\mu_1(\omega_1) \right] d\mu_2(\omega_2) = \int \left[\int f(\omega_1, \omega_2) d\mu_2(\omega_2) \right] d\mu_1(\omega_1)\,.$$

7.2 Independence

Definition 7.11 (Independence between collection of sets) *Let $(\Omega, \mathcal{F}, P)$ be a probability space. For two collections $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{F}$, we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are **independent** if $P(A_1 \cap A_2) = P(A_1)P(A_2)$ for all $A_1 \in \mathcal{C}_1$, $A_2 \in \mathcal{C}_2$.*

Definition 7.12 (Independence between random variables) *Let $(\Omega, \mathcal{F}, P)$ be a probability space. For $i = 1, 2$ let $(S_i, \mathcal{A}_i)$ be measurable spaces and $X_i : \Omega \mapsto S_i$ be $\mathcal{F}/\mathcal{A}_i$ measurable functions. (Hence X_1 and X_2 are random variables.) Let $\sigma(X_i)$ be the σ-field $X_i^{-1}(\mathcal{A}_i) \subset \mathcal{F}$ generated by function X_i. We say that X_1 and X_2 are **independent** if $\sigma(X_1)$ and $\sigma(X_2)$ are independent collections.*

Theorem 7.13 *Let X_1, X_2 be two random variables following the definition above. Define another random variable $X : \Omega \mapsto S_1 \times S_2$ by $X = (X_1, X_2)$. Then its distribution μ_X (induced measure on $(S_1 \times S_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$) is the product measure $\mu_{X_1} \times \mu_{X_2}$ if and only if X_1 and X_2 are independent.*

Proof: By definition X_1 and X_2 are independent if and only if for all $B_1 \in X_1^{-1}(\mathcal{A}_1), B_2 \in X_2^{-1}(\mathcal{A}_2)$ we have $P(B_1 \cap B_2) = P(B_1)P(B_2)$. It remains to show that $\mu_X = \mu_{X_1} \times \mu_{X_2}$ if and only if $\forall B_1 \in X_1^{-1}(\mathcal{A}_1), B_2 \in X_2^{-1}(\mathcal{A}_2), P(B_1 \cap B_2) = P(B_1)P(B_2)$.

Proof of if. Suppose $\forall B_1 \in X_1^{-1}(\mathcal{A}_1), B_2 \in X_2^{-1}(\mathcal{A}_2), P(B_1 \cap B_2) = P(B_1)P(B_2)$ holds. For any $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$, we have

$$\begin{aligned} \mu_X(A_1 \times A_2) &= P\left(\{\omega \in \Omega : X_1(\omega) \in A_1, X_2(\omega) \in A_2\}\right) = P\left(X_1^{-1}(A_1) \cap X_2^{-1}(A_2)\right) \\ &= P(X_1^{-1}(A_1))P(X_2^{-1}(A_2)) = \mu_{X_1}(A_1)\mu_{X_2}(A_2)\,. \end{aligned}$$

Therefore, $\mu_X = \mu_{X_1} \times \mu_{X_2}$.

Proof of only if. Suppose $\mu_X = \mu_{X_1} \times \mu_{X_2}$. Then for all $B_1 \in X_1^{-1}(\mathcal{A}_1), B_2 \in X_2^{-1}(\mathcal{A}_2)$,

$$\begin{aligned}
P(B_1 \cap B_2) &= P(X_1^{-1}(X_1(B_1)) \cap X_2^{-1}(X_2(B_2)) = P(X^{-1}(X_1(B_1) \times X_2(B_2))) \\
&= \mu_X(X_1(B_1) \times X_2(B_2)) = \mu_{X_1}(X_1(B_1))\mu_{X_2}(X_2(B_2)) = P(B_1)P(B_2)\,.
\end{aligned}$$

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7.3 Stochastic Processes

Definition 7.14 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and T be a set. For each $t \in T$, there is a measurable space $(\mathcal{X}_t, \mathcal{F}_t)$ and a random variable $X_t : \Omega \mapsto \mathcal{X}_t$. The collection $\{X_t : t \in T\}$ is called a* ***stochastic process****, and T is called the* ***index set****.*

Example 7.15 *Let $T = \{1, ..., k\}$. A vector of random variables $X = [X_1, ..., X_k]$ is a stochastic process.*

Example 7.16 (Random probability measure) *Let $\Theta : \Omega \mapsto \mathbb{R}$ be a random variable, $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be a nonnegative function such that $\int_{\mathbb{R}} f(x, \theta)dx = 1$ for all $\theta \in \mathbb{R}$. For example, $f(x, \theta) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\theta)^2}{2}\right)$. Let $T = \mathcal{B}$. For each $B \in \mathcal{B}$ consider random variable $X_B : \Omega \mapsto R$ defined by $X_B(\omega) = \int_B f(x, \Theta(\omega))dx$. Then the stochastic process $\{X_B : B \in \mathcal{B}\}$ is a random probability measure.*

Example 7.17 (Empirical measure) *Let $X_1, ..., X_n$ be i.i.d. samples from some P on $\mathbb{R}$. Define the empirical measure P_n on $(\mathbb{R}, \mathcal{B})$ as $P_n(B) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}\{X_i \in B\}$ for all $B \in \mathcal{B}$. (Why introduced here?)*

Remark 7.18 *The product set $\mathcal{X} = \prod_{t \in T} \mathcal{X}_t$ can be viewed as the set of all functions $f : T \mapsto \bigcup_{t \in T} \mathcal{X}_t$ such that $f(t) \in \mathcal{X}_t$ for all $t \in T$. For example, when $\mathcal{X}_t = \mathcal{Y}$ for all t, $\mathcal{X} = \prod_{t \in T} \mathcal{X}_t = \mathcal{Y}^T$ is the set of all functions from T to $\mathcal{Y}$. In a stochastic process, the random variable $X : \Omega \mapsto \mathcal{X}$ defined by $X(\omega) = \{X_t(\omega) : t \in T\}$ induces a probability distribution over $\mathcal{X} = \prod_{t \in T} \mathcal{X}_t$, i.e. over all functions $f : T \mapsto \bigcup_{t \in T} \mathcal{X}_t$ such that $f(t) \in \mathcal{X}_t$ for all $t \in T$.*