36-752: Advanced Probability	**Spring 2018**

Lecture 3: February 6

Lecturer: Alessandro Rinaldo *Scribe: Ron Yurko*

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

Last Time:

- Measure (Probability is a special case)
- Measurable space
- Satisfying properties → notably countable additivity
- Inclusion-Exclusion Principle (see Wikipedia):
  if $\mu(\bigcup_{i=1}^{n} A_i) < \infty$ then,

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\leq i_1<\ldots<i_k=n} \mu(A_{i_1} \cap \ldots \cap A_{i_n})$$
$$= \sum_{J\subseteq[n]=\{1,\ldots,n\}} (-1)^{|J|-1}\mu(\bigcup_{i\in J} A_i), \text{ where } J \neq 0$$

- $\pi - \lambda$ Theorem (see notes for proof)
  - uniqueness of measure
  - 2 measures on σ-field describing properties over π-system is unique
  - way of describing uniqueness
  - more of a technicality

New Material:

3.1 Caratheodory's Extension

Theorem 3.1 *Let μ be a σ-finite measure over a field $\mathcal{C}$ (need to assume we have countable additive sets over the field). Then μ has a unique extension to a measure on $\sigma(\mathcal{C})$.*

See notes for proof.

3.2 Measures from CDFs

Let $F : \mathbb{R} \to [0, 1]$ be a CDF such that:

1. $\underset{x\to\infty}{lim} F(x) = 1,\ \underset{x\to\infty}{lim} F(x) = 0$
2. F has left limits: $\underset{y\uparrow x}{lim} F(y)$ exists $\forall x \in \mathbb{R}$
3. Continuity from the right: $\underset{y\downarrow x}{lim} F(y) = F(x)$
   Functions of this form are known as Càdlàg (see Wikipedia).
4. Monotonicity: $x \leq y \Rightarrow F(x) \leq F(y)$
   Let $\mathbf{C} = \{$set of discontinuity of Càdlàg function$\}$, then $\mathbf{C}$ is countable! (Any probability distribution defined over discrete space $\to$ countably infinite set of discontinuity).

Let x be a point of discontinuity for F,

$$\underset{y\uparrow x}{lim} F(y) = F(x^-) < F(x) = \underset{y\downarrow x}{lim} F(y)$$

Let r_x be any rational such that

$$F(x^-) < r_x \leq F(x)$$
$$\Rightarrow \mathbf{C} \text{ is a one-to-one to a subset of } \mathbb{Q}$$
$$\Rightarrow \mathbf{C} \text{ is countable}$$

Recall the field $\mathcal{U}$ of finite disjoint unions of sets of the form $(a, b]$, $-\infty \leq a < b$ and complements.
Let $\mu : \mathcal{U} \Rightarrow [0, 1]$ given by,

$$\mu(A) = \sum_{k=1}^{n} F(b_k) - F(a_k)$$

when $A = \bigcup_{k=1}^{n}(a_k, b_k]$ disjoint!
If μ is a measure on $\mathcal{U}$ then it will extend to unique measure on $\sigma(\mathcal{U}) = \mathcal{B}$, Borel σ-field.
We need to show that μ is countably additive over $\mathcal{U}$.

Lemma 3.2 *Let $(a, b] = \bigcup_{k=1}^{\infty}(c_k, d_k]$ (infinite disjoint union) then,*

$$\begin{aligned}\mu((a, b]) &= F(b) - F(a) \\ &= \sum_{k=1}^{\infty} F(d_k) - F(c_k)\end{aligned}$$

Proof: Since $(a, b] \supseteq \bigcup_{k=1}^{n}(a_k, b_k]$ (finite n),

$$\Rightarrow F(b) - F(a) \geq \sum_{k=1}^{n} F(d_k) - F(c_k),\ \forall n$$

because sets are disjoint and by finite additivity.
To show for infinite, take limit on both sides,

$$\Rightarrow F(b) - F(a) \geq lim \sum_{k=1}^{n} F(d_k) - F(c_k) = \sum_{k=1}^{\infty} F(d_k) - F(c_k)$$

Assume a, b are finite. For each $k \ \exists \ e_k > d_k$ such that,

$$F(d_k) \leq F(e_k) \leq F(d_k) + \frac{\epsilon}{2^n}$$

by right continuity, for some arbitrarily small ϵ. Also $\exists \ f > a \ s.t. \ F(a) \geq F(f) - \epsilon$, holds by right continuity.

Next the set $[f, b]$,
If $\mathcal{C}$ is compact and $\mathcal{C} \subset \bigcup_\alpha \bigcup_\alpha$ (open sets) then $\exists \ a$ finite collection $\alpha_1, \ldots, \alpha_n \ s.t. \ \mathcal{C} \subset \bigcup_i^n \bigcup_{\alpha_i}$.
In $\mathbb{R}^k$, $\mathcal{C}$ is compact iff it is closed and bounded.
Since $[f, b] \subseteq \bigcup_{k=1}^{\infty} (c_k, e_k)$ there exist finitely many indices k, which assume to be the first n such that:

$$\begin{aligned}
[f, b] \subseteq \bigcup_{k=1}^{n} (c_k, e_k) \subset \bigcup_{k=1}^{n} (c_k, e_k] \\
F(b) - F(a) &\leq F(b) - F(f) + \epsilon \\
&\leq \epsilon + \sum_{k=1}^{n} F(e_k) - F(c_k) \\
&\leq \epsilon + \sum^{n} [(F(d_k) - F(c_k)) + \frac{\epsilon}{2^k}] \\
&\leq \epsilon + \sum_{k}^{n} \frac{\epsilon}{2^k} + \sum_{k}^{n} F(d_k) - F(c_k)
\end{aligned}$$

$\forall \ \epsilon$ and all n.
Take limit as $n \to \infty$,

$$= \epsilon + \epsilon + \sum_{k=1}^{\infty} F(d_k) - F(c_k)$$

Since ϵ is arbitrarily small,

$$F(b) - F(a) = \sum_{k=1}^{n} F(d_k) - F(c_k)$$

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(One of the purposes of seeing this proof is to see the use of compact sets. This is how you construct measures on Borel σ-field, **HW** - if you have a measure on a Borel σ-field then you also have a CDF, not hard to show).
This result holds even if F is unbounded. If $F(x) = x$ then the resulting μ on $(\mathbb{R}, \mathcal{B})$ is the Lebesgue measure.

If I is an interval $(a, b]$ or $[a, b)$ or (a, b) (any form) then $\mu(I) =$ length of interval $b - a$ (over real, measure of $\mathcal{B}$ σ-field defining length) then it is σ-finite.

3.3 Extension to $\mathbb{R}^k$

Let $F : \mathbb{R}^k \to [0, 1] \ s.t.$,

1. F is non-decreasing (if $x \leq y$ in $\mathbb{R}^k, x_i \leq y_i \ \forall \ i, \Rightarrow F(x) \leq F(y)$).
2. F has left limits
3. F is right-continuous
4. $$\lim_{x \to +\infty} F(x) = 1$$ $$\lim_{x \to -\infty} F(x) = 0$$

Let A be a hyper-rectangle,

$$A = (a, b] \times \ldots \times (a_k, b_k]$$

$\mathcal{B}^k$ is a σ-field generated by such sets.

Example: $k = 2$,

$$\mu((a_1, b_1] \times (a_2, b_2]) = F(b_1, b_2) - F(a_1, b_2) - F(a_2, b_1) + F(a_1, a_2)$$

where $F(b_1, b_2) = \mu((-\infty, b_1] \times (-\infty, b_2])$.
But μ is not a measure,

$$F(x_1, x_2) = \begin{cases} 1, & \text{if } x_1, x_2 \geq 1 \\ \frac{2}{3}, & \text{if } x_1 \geq 1, x_2 \in [0, 1) \\ \frac{2}{3}, & \text{if } x_2 \geq 1, x_1 \in [0, 1) \\ 0, & \text{otherwise} \end{cases}$$

$$\mu(\{1, 1\}) = \frac{-1}{3}$$
$$a_1 = a_2 = 1 - \epsilon$$
$$b_1 = b_2 = 1$$

Let $\epsilon \downarrow 0$, as we increase dimensions it no longer holds.
Leads to one more condition,

5. Let $A = (a_1, b_1] \times \ldots \times (a_k, b_k]$, and $V_A = \{a_1, b_1\} \times \ldots \times \{a_k, b_k\}$ equal the vertices of A. For $v \in V_A$ let $sign(v) = (-1)^{\# \ a's \ in \ v}$., let $\Delta_A F = \sum_{v \in V_A} sign(v) F(v)$, condition is that $\Delta_A F \geq 0$ for all such A.

This is to make us aware of the difficulties of switching to higher dimensions and how we get Lebesgue measure on $\mathbb{R}^k$. Apply Caratheodory's extension $\Rightarrow$ Lebesgue measure on $\mathbb{R}^k$, $\mu(A) =$ volume of A.

3.4 Measurable Functions

Let $(\Omega, \mathcal{F}, P)$ be a probability space (where P is a probability measure and $f : \Omega \to S$.
What is $P(h(\omega) \in A), \ A \subseteq S$ (also equals $P(\omega \ : \ h(\omega) \in A)$)? Write as "$Pr(\{\omega \ : \ h(\omega) \in A\})$".

Definition 3.3 *Let* $(\Omega, \mathcal{F})$ *and* $(S, \mathcal{A})$ *be measurable spaces.*
Let $f : \Omega \to S$ *s.t.* $f^{-1}(A) \in \mathcal{F}\ \forall A \in \mathcal{A}$ *then* f *is* $\mathcal{F}/\mathcal{A}$ *measurable or measurable.*

Example: $\mathcal{F} = 2^{\Omega}$ then every $f : \Omega \to S$ is measurable.

Example: $\mathcal{A} = \{\emptyset, S\}$ then any $f : \Omega \to S$ is measurable.

$f(B) \in \mathcal{A},\ \forall B \in \mathcal{B}$? $\to$ is this the case?
$\to$ even if f is measurable there is no guarantee that $f(B) \in \mathcal{A}$, example is $\mathcal{A} = \{\emptyset, S\}$.

Facts:

- $f^{-1}(A^c) = (f^{-1}(A))^c)$
- $f^{-1}(\bigcup_n A_n) = \bigcup_n f^{-1}(A_n)$
- $f^{-1}(\bigcap_n A_n) = \bigcap_n f^{-1}(A_n)$

In general $f(B^c) \neq (f(B))^c$ as these are very different sets.