36-752: Advanced Probability **Spring 2018**

Lecture 4: February 8

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4.1 Measurable Functions

Recall the definition of a measurable function from last lecture.

Definition 4.1 *Let $(\Omega, \mathcal{F})$ and $(S, \mathcal{A})$ be measurable spaces. A function $f : \Omega \to S$ is* **measurable** *when $\forall A \in \mathcal{A}$,*

$$f^{-1}(A) = \{\omega \in \Omega : f(\omega) \in A\} \in \mathcal{F}$$

Remark 4.2 *If f is measurable, this does not imply $f(B) \in \mathcal{A}, \forall B \in \mathcal{F}$.*

Example: Consider $\mathcal{A} = \{\emptyset, S\}$, where $\{s \in S : \exists \omega \in B, f(\omega) = s\}$ and $f(\omega) = a$, $\forall \omega \in \Omega$. Then $B \in \mathcal{F}$, and $B \neq \Omega$, and $f(B) \notin A$. On the homework, you will prove $f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\}$ is a σ-field on Ω, using properties of images.

Just like a σ-field can be generated by a set, it can also be generated by a measurable function.

Definition 4.3 *The σ-field $f^{-1}(\mathcal{A})$ is called the* **σ-field generated by** *f. It is the smallest σ-field on Ω such that f is measurable.*

Here is a useful result, which - as we will see later - begets a number of useful consequences for the measurability of continuous, monotone, and multivariate functions.

Lemma 4.4 *Let $(\Omega, \mathcal{F})$ and $(S, \mathcal{A})$ be measurable spaces, and let $f : \Omega \to S$ and let C be a collection of subsets of A generating $\mathcal{A}$, i.e. $\sigma(C) = \mathcal{A}$. Then f is measurable if and only if $f^{-1}(C) \subseteq \mathcal{F}$.*

In other words, if the pre-image of every set is measurable, then f is measurable. Note, it should be enough to check the condition on the σ-field of the pre-image. Here we prove the *only if* direction because it is more straightforward.

Proof: Let $\mathcal{A}' = \{A \subseteq S : f^{-1}(A) \in \mathcal{F}\}$. Then you can show $\mathcal{A}'$ is a σ-field over S and contains C. $\mathcal{A} = \sigma(C) \subseteq \mathcal{A}'$, the smallest σ-field containing C. The result follows because if f is $\mathcal{F}/\mathcal{A}'$-measurable, it is also $\mathcal{F}/\mathcal{A}$-measurable for $\mathcal{A} \subseteq \mathcal{A}'$. (A σ-field larger than the trivial σ-field.) ■

4.1.1 Consequences of Lemma 4.5

There are a number of consequences of the previous lemma, where we saw that $f : \Omega \to S$ is measurable if and only if $f^{-1}(C) \subseteq \mathcal{F}$, where $\sigma(C) = \mathcal{A}$.

1. Continuous functions* are measurable.
   **when they are endowed with Borel σ-fields.*

   In particular, if X and Y are topological spaces (open and closed sets are well defined) then $f : X \to Y$ is continuous if $f^{-1}(U)$ is an open set in X whenever U is an open set in Y.

2. Monotone functions are measurable.

   Let $f : \Omega \to \mathbb{R}$, where Ω has σ-field $\mathcal{F}$ and $\mathbb{R}$ has Borel σ-field $\mathcal{B} = \sigma(\{(-\infty, b), b \in \mathbb{R}\})$, then f is measurable if $f^{-1}((-\infty, b))$ is measurable for all $b \in \mathbb{R}$.

   Dealing with $\overline{\mathbb{R}}$ is trickier. Note, if $f : \Omega \to \overline{\mathbb{R}}$ where $\overline{\mathbb{R}}$ is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ then $f^{-1}(\{\infty\}) = \cap_n \{\omega : f(\omega) > n\}$.

3. Multivariate functions can be measurable.

   Let f be a multivariate function, $f : \Omega \to \mathbb{R}^k$ where $\mathbb{R}^k$ has the σ-field $\mathcal{B}^k = \sigma(\{(a, b] \times \cdots \times (a_n, b_n]\})$ and $a_i < b_i$ for all i, then f is measurable if $f^{-1}((a, b] \times \cdots \times (a_n, b_n]) \in \mathcal{F}$ for all $a_i < b_i$. Where $f^{-1}((a, b] \times \cdots \times (a_n, b_n])$ is
   $$\{\omega \in \Omega, f_i(\omega) \in (a_i, b_i], \forall\ i\}$$
   and notice this is equivalent to
   $$\bigcap_{i=1}^{k} \{\omega : f_i(\omega) \in (a_i, b_i]\} = \bigcap_{i=1}^{k} f_i^{-1}((a_i, b_i])$$
   which shows that f is measurable if and only if each f_i is measurable.

To wrap up our study of measurable functions, here are some non-trivial examples and proofs of measurability.

Theorem 4.5 *Let $f_n : \Omega \to \mathbb{R}$ be measurable for all n. Then*

(i) If $\omega \to \limsup_n f_n(\omega)$ then $\limsup_n f_n$ and $\liminf_n f_n$ are measurable.

(ii) $\{\omega : \lim_{n\to\infty} f_n(\omega)$ exists$\}$ is a measurable set.

(iii) Let $f = \lim_n f_n$. If $f_n(\omega) = f$ for all ω, then f is measurable.

Proof: Part (i). We know $\limsup_n f_n(\omega) = \lim \sup_{k \geq n} f_k(\omega)$, but for any $b \in \mathbb{R}$:

$$\{\omega : \sup_{k\geq n} f_k(\omega) \leq b\} = \bigcap_{k\geq n} \{\omega : f_k(\omega) \leq b\}$$

and a countable intersection of measurable sets is measurable. ■

Proof: Part (ii). Homework! ■

Proof: Part (*iii*). Consider $\{\omega : f(\omega) > b\}$, and $b \in \mathbb{R}$ is arbitrary. All we know is f_n are measurable, so we must express in terms of that, which is equivalent to:

$$\begin{aligned}
\{\omega : \lim_{n\to\infty} f_n(\omega) > b\} &= \bigcup_{r=1}^{\infty} \liminf_n \left\{\omega : f_n(\omega) > b + \frac{1}{r}\right\} \\
&= \bigcup_{r=1}^{\infty} \left\{\omega : f_n(\omega) > b + \frac{1}{r} \text{ eventually}\right\} \\
&= \bigcup_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{\omega : f_k(\omega) > b + \frac{1}{r}\right\}
\end{aligned}$$

and since $\{\omega : f_k(\omega) > b + \frac{1}{r}\}$ is measurable and we are taking a finite number of countable set operations, that must be measurable and so f must be measurable. ■

4.2 Random Variables and Induced Measure

Definition 4.6 *Let $X : \Omega \to \overline{\mathbb{R}}$ and let $(\Omega, \mathcal{F}, P)$ be a probability space (a measure that happens to be a probability). Then X is a* **random variable***.*

Example: Let $\Omega = [0,1]$, $\mathcal{F}$, $\mathcal{B}$ is restricted to $[0,1]$, and P is a Lebesgue measure on $[0,1]$. $X(\omega) = \lfloor 2\omega \rfloor$ and $Z(\omega) = \omega$, then X takes values in $[0,1]$, so $X(\omega) = \mathbb{I}_{[0,\frac{1}{2})}(\omega)$. Almost everywhere, $\mu(X^{-1}(\{0\})) = \mu(X^{-1}(\{-1\})) = \frac{1}{2}$ and this is a Bernoulli random variable.

Recognize that $\mu(Z^{-1}([0,c))) = c$ for $c \in [0,1]$ is a Uniform random variable.

Lemma 4.7 *Let $(\Omega, \mathcal{F}, \mu)$ and $(S, \mathcal{A})$ be measure spaces and $f : \Omega \to S$ be measurable, then f* **induces a measure** *ν on $(S, \mathcal{A})$, which is given by $\nu(\mathcal{A}) = \mu(f^{-1}(A))$ for all $A \in \mathcal{A}$.*

Definition 4.8 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and X be a random variable. The measure on $(\mathbb{R}, \mathcal{B})$ induced by P and X is the* **probability distribution** *of X.*

Notice that $Pr(X \in A), A \in \mathcal{B}$, a sufficiently well-balanced set, $\exists$ $(\Omega, \mathcal{F}, P)$ and X is measurable. $Pr(X \in A) = P(\{\omega \in \Omega : X(\omega) \in A\}) = P(X^{-1}(A)) = \mu_X(A)$, the probability distribution of X.

4.2.1 Measures Acting on Functions

This section is covered in section 3 of lecture notes 2. Our goal is to examine integrals

$$\int_\Omega f d\mu$$

or rather, how measures act on functions.

Example: Some examples are already familiar.

- If μ is the Lebesgue measure, then $d\mu = dx$.
- If μ is a PDF and f is the identity function, then $\int_\Omega f d\mu$ is the expectation.

Definition 4.9 *A* **simple function** *is a function taking finitely many values. Let $(\Omega, \mathcal{F})$ be a measurable space, and let f be a simple function taking values $\{a_1 \cdots a_n\}$ (distinct reals). The canonical form of f is:*

$$f(\omega) = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}(\omega)$$

where $\mathbb{I}_{A_i}(\omega)$ is 1 when $\omega \in A$ and 0 when $\omega \notin A$, and also $A_i = f^{-1}(a_i) = \{\omega : f(\omega) = a_i\}$ and $A_1 \cdots A_n$ is a measurable partition of Ω.

Theorem 4.10 *Let f be a non-negative measurable function, then there exists $\{f_n\}_n$ sequence of non-negative simple functions such that $f_n(\omega) \in f(\omega)$ for all $\omega \in \Omega$ (point-wise convergence).*

Example: Consider

$$f_n(\omega) = \begin{cases} \frac{k-1}{2^n} & \frac{k-1}{2^n} \leq f(\omega) < \frac{k}{2^n} \\ n & f(\omega) \geq n \end{cases}$$

where $n = 1 \cdots n2^n$. Notice that in the first condition, it approaches $f(\omega)$ as $n \to \infty$. Now

$$f = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbb{I}_{A_k}(\omega) + n\mathbb{I}_{A_{\infty,n}}(\omega)$$

and so, if f is real-valued then

$$f(\omega) = f^+(\omega) - f^-(\omega)$$

where $f^+ \geq 0$ and $f^- \geq 0$, and $f^+(\omega) = \max\{f(\omega), 0\}$ and $f^-(\omega) = -\min(\{f(\omega), 0\}$.