36-752: Advanced Probability Overview **Spring 2018**

Lecture 1: January 30

Lecturer: Alessandro Rinaldo *Scribes: Natalia Lombardi de Oliveira*

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Overview: In this lecture, we present some basic concepts that will be used systematically throughout the course. We start by presenting definitions and notation on set theory and then move on to defining fields and σ-fields.

Set theory:

Basic notation:

- Ω : universal set
- $A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$
- $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$
- $A^C = \{\omega : \omega \notin A\}$
- $A \subseteq B = \{\omega : \omega \in A \Rightarrow \omega \in B\}$
- $A \setminus B = A \cap B^C$
- $A \triangle B = (A \setminus B) \cup (B \setminus A)$
- $A \times B = \{(a, b) : a \in A, b \in B\}$
- $A^k = A \times A \times \ldots \times A$ k times

De Morgan's Law:

- $(\bigcup\limits_n A_n)^C = \bigcap\limits_n A_n^C$
- $(\bigcap\limits_n A_n)^C = \bigcup\limits_n A_n^C$

Definition 1.1 *Let $A_1, A_2, \ldots$ be a sequence of sets (finite or infinite). The sequence is monotone increasing if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$. We define $A = \bigcup\limits_n A_n$ and the limit $An \uparrow A$. The sequence is monotone decreasing if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$. We define $A = \bigcap\limits_n A_n$ and the limit $An \downarrow A$.*

Example: $a, b \in \mathbb{R}, a < b$

- $A_n = [a + \frac{1}{n}, b - \frac{1}{n}]$
  $\bigcup\limits_n A_n = (a, b)$.

- $A_n = (a - \frac{1}{n}, b + \frac{1}{n})$
  $\bigcap_n A_n = [a, b]$.

Definition 1.2 *Let $A_1, A_2, \ldots$ be an arbitrary sequence of sets. The limit superior is*

$$\limsup_n A_n = \bigcap_n^{\infty} \bigcup_{k=n}^{\infty} A_k,$$

that is, $x \in \limsup_n A_n$ if and only if $\forall n, \exists k \geq n$ such that $x \in A_k$. Intuitively, x belongs to infinitely many A_n's or $x \in A_n$ infinitely often.

Definition 1.3 *Let $A_1, A_2, \ldots$ be an arbitrary sequence of sets. The limit inferior is*

$$\liminf_n A_n = \bigcup_n^{\infty} \bigcap_{k=n}^{\infty} A_k,$$

that is, $x \in \liminf_n A_n$ if and only if $\exists n$ such that $x \in A_k \forall k \geq n$. $\forall n, \exists k \geq n$ such that $x \in A_k$. Intuitively, x belongs to all but finitelly many A_n's or $x \in A_n$ eventually.

Result: If $\{A_n\}_n$ is monotone, then $\lim_n A_n = \limsup_n A_n = \liminf_n A_n$.

Example: Let (a, b) and (c, d) be disjoint intervals of $\mathbb{R}$.

$$A_n = \begin{cases} (a, b), & \text{if } n \text{ is odd,} \\ (c, d), & \text{if } n \text{ is even.} \end{cases}$$

Then,

$$\begin{aligned} \limsup_n A_n &= (a, b) \cup (c, d) \\ \liminf_n A_n &= \emptyset. \end{aligned}$$

Definition 1.4 *The set A is finite if $|A| < \infty$. The set A is infinite if $|A| = \infty$.*

Definition 1.5 *The set A is countable if $\exists \phi$ such that $\phi : A \to \mathbb{N}$ and ϕ is injective. If a set is not countable, it is uncountable. A countable set can be finite or infinite.*

Result: If $\{A_n\}_n$ is countable, then $\bigcup_n A_n$ is countable.

Proof: First, notice that there is a bijection between $\bigcup_n A_n$ and $\mathbb{N}^2$.

$$\begin{array}{cccc} A_1 & A_2 & A_3 & \ldots \\ a_{11} & a_{12} & a_{13} & \ldots \\ a_{21} & a_{22} & a_{23} & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

Now, to show that $\mathbb{N}^2$ is countable, pick p, q prime numbers. Define $\phi : \mathbb{N}^2 \to \mathbb{N}$ such that $(m, n) \to p^m q^n$. Notice that if $(m, n) \neq (m', n')$, then $p^m q^n \neq p^{m'} q^{n'}$. Therefore, ϕ is injective.

Result: $\{A_n\}_n$ countable does not imply $\prod_n A_n$ countable.

Proof: Counter example: Assume that $0, 1^\infty$ is countable. Then, we can write it as $s^{(1)}, s^{(2)}, s^{(3)}, \ldots$, where each $s^{(i)}$ is an infinite binary sequence.

Let s be an infinite binary sequence considered as follows:

$$s_n = |1 - s_n^{(n)}|.$$

Then, notice that $s \neq s^{(n)} \forall n$. Therefore, $0, 1^\infty$ is not countable.

Result: The set $(0, 1]$ is uncountable.

Proof: Since each number $x \in (0, 1]$ can be expressed as an infinite binary sequence of zeros and ones that does not terminate with zeros, then it is uncountable.

σ-fields:

Definition 1.6 *Let Ω be a set. A collection $\mathcal{F}$ of subsets of Ω is called a field if*

- $\Omega \in \mathcal{F}$
- $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
- *if* $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

$\mathcal{F}$ is a σ-field if, in addition,

- *if a sequence of sets* $\{A_n\}_n \in \mathcal{F}$, *then* $\bigcup_n A_n \in \mathcal{F}$.

Definition 1.7 *A measurable space is a pair* $(\Omega, \mathcal{F})$.

Example:

- Trivial σ-field: $\{\Omega, \emptyset\}$,
- largest σ-field: 2^Ω (power set).

Properties of σ-fields: If $\{\mathcal{F}_n\}_n$ is a sequence of σ-fields:

- $\bigcap_n \mathcal{F}_n$ is a σ-field,
- $\bigcup_n \mathcal{F}_n$ is not (in general) a σ-field.

Example: Let $\Omega = \mathbb{N}$ and $\mathcal{F}_n$ be the σ-field of all subsets of $\{1, 2, \ldots, n\}$. Notice that for $i = 1, 2, 3, \ldots$, $\{2i\} \in \bigcup_n \mathcal{F}_n$, but $\bigcup_i \{2i\} \notin \bigcup_n \mathcal{F}_n$.

Definition 1.8 *Let $\mathcal{A}$ be a collection of subsets of Ω. The σ-field generated by $\mathcal{A}$ is the smallest sigma-field that contains $\mathcal{A}$ and is denoted by $\sigma(\mathcal{A})$.*

Example: $\sigma(A) = \{\Omega, \emptyset, A, A^C\}$.