36-752: Advanced Probability Overview	**Spring 2018**

Lecture 16: March 29

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Strong Law of Large Numbers (SLLN):

Lemma 16.1 *(Kronecker's lemma)*

Let $\{x_n\}_n, \{a_n\}_n, a_i > 0, i = 1, 2, \ldots$ *and* $a_n \to \infty$. *Suppose* $\sum_{n=1}^{\infty} \frac{x_n}{a_n} < \infty$. *Then,* $\frac{\sum_{i=1}^{n} x_i}{a_n} \to 0$.

Corollary 16.2 $X_1, X_2, \ldots$ *are independent RV's with* $E[X_i] = 0$ *all* i. *If* $\sum_{i=1}^{\infty} \frac{E[X_i^2]}{a_i^2} < \infty$ *then by* L_2*/a.s. convergence result,*

$$\frac{\sum_{i=1}^{n} X_i}{a_n} \xrightarrow{a.s.} 0.$$

Then, if $Var[X_i] = \sigma^2$ *all* i, *take* $a_n = n$ *so that* $\sum_{i=1}^{\infty} \frac{\sigma^2]}{n^2} < \infty \implies \frac{S_n}{n} \xrightarrow{a.s.} 0$, $S_n = \sum_{i=1}^{n} X_i$.

This remains true even if $a_n = n^{1/2+\epsilon}, \epsilon > 0$, *then*

$$\sum_{i=1}^{\infty} \frac{\sigma^2]}{n^{1+2\epsilon}} < \infty \implies \frac{S_n}{n^{1/2+\epsilon}} \xrightarrow{a.s.} 0.$$

Theorem 16.3 *(Kolmogorov's SLLN)*

Let $X_1, X_2, \ldots$ *be a sequence of iid RV's such that* $E[|X_i|] < \infty$, *all* i. *Then,*

$$\frac{S_n}{n} \xrightarrow{a.s.} E[X_1].$$

Proof: By truncation argument.

Law of iterated logarithms:

Let $X_1, X_2, \ldots$ be a sequence of iid RV's such that $E[X_i] = 0, Var[X_i] = \sigma^2$, all i. We know that $\frac{S_n}{n^{1/2+\epsilon}} \xrightarrow{a.s.} 0, \epsilon > 0$. Then,

$$\limsup_n \frac{S_n}{\sigma\sqrt{nlog(log(n))}} \xrightarrow{a.s.} \sqrt{2} \text{ and } \liminf_n \frac{S_n}{\sigma\sqrt{nlog(log(n))}} \xrightarrow{a.s.} -\sqrt{2}.$$

We know that $\limsup_n x_n = \inf_n sup_{k \geq n} x_k$. If $x = \limsup_n x_n$ then $\forall \epsilon > 0, \exists N(\epsilon)$ such that $x_n \leq x + \epsilon, \forall n > N(\epsilon)$. In particular, $S_n \geq \sigma\sqrt{nlog(log(n))} + \epsilon$ i. o. $\forall \epsilon > 0$ small enough.

Some interesting results:

- $\limsup_n \frac{S_n}{\sqrt{n}} = \infty$
- $\liminf_n \frac{S_n}{\sqrt{n}} = -\infty$
- $\frac{S_n}{\sqrt{n}} \approx N(0, \sigma^2)$ by CLT
- $\frac{S_n}{\sqrt{2nlog(log(n))}} \xrightarrow{p} 0$, which means that for every ϵ small enough, $Z_n = \frac{S_n}{\sqrt{2nlog(log(n))}}$ will be in $(-\epsilon, \epsilon)$ with high probability, but $Z_n \in (\sqrt{2} - \epsilon, \sqrt{2})$ i.o. Recall that almost sure convergence is a statement about individual ω's. The set of ω's for which Z_n goes in or out of $(-\epsilon, \epsilon)$ or $(\sqrt{2} - \epsilon, \sqrt{2})$ changes with n.

Convergence in distribution:

Convergence in distribution can also be called *w*eak convergence or *w*eak+ convergence.

Let $(\mathscr{X}, d)$ with Borel σ-field $\mathcal{B}$. Let $\{X_n\}, X$ taking values in $\mathscr{X}$ ($\{X_n\}$ and X do not need to be defined over the same probability space).

Definition 16.4 *(Convergence in distribution)*

$X_n \xrightarrow{d} X$ (or $X_n \xrightarrow{\mathcal{D}} X$ or $X_n \rightsquigarrow X$ or $X_n \Rightarrow X$) when $\lim_{n\to\infty} E[f(X_n)] = E[f(X)]$ for all continuous bounded functions f (or Lipschitz bounded functions).

Let $Z_1, Z_2, \ldots$ independent $N(0,1)$ and $X_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i$ then $X_n \sim N(0,1) \implies X_n \xrightarrow{d} X, X \sim N(0,1)$.

Equivalently, if $X_n \sim N(0,1)$, independent for all n, then X_n does not converge to anything.

- Convergence in distribution is a statement about distributions of X's.
- If $X_1 \sim N(0,1), X_2 \sim N(0,1)$, X_1, X_2 independent, then $X_1 \neq X_2$ almost surely but $X_1 = X_2$ in distribution.

Example: $\{X_n\}$ a sequence of Bernoulli R.V.'s: $X_n \sim Bernoulli(\frac{1}{2}\frac{n+1}{n})$, $X \sim Bernoulli(1/2)$. Note that $X_n \xrightarrow{d} X$, but X_n may or may not $\xrightarrow{p} X$; it depends on the dependency structure.

Theorem 16.5 *(Portmanteau Thm)*

The following conditions are equivalent:

1. *$E[f(X_n)] \to E[f(X)]$ all bounded continuous f's.*
2. *$\forall C \subseteq \mathscr{X}$ closed, $\limsup_n P(X_n \in C) \leq P(X \in C)$.*
3. *$\forall A \subset \mathscr{X}$ open, $\liminf_n P(X_n \in A) \geq P(X \in A)$.*
4. *$\forall B \in \mathcal{B}$ such that $P(X \in \partial B) = 0$, $\lim_n P(X_n \in B) = P(X \in B)$.*

Proposition 16.6

1. *$X_n \xrightarrow{p} X \implies X_n \xrightarrow{d} X$.*
2. *If X is degenerate ($P(X = c) = 1$, some $c \in \mathscr{X}$) and $X_n \xrightarrow{d} X$, then $X_n \xrightarrow{p} X$.*