36-752: Advanced Probability **Spring 2018**

Lecture 24: May 1

Lecturer: Alessandro Rinaldo *Scribes: Yifan Wu*

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

This lecture's notes illustrate some uses of various LaTeX macros. Take a look at this and imitate.

24.1 Concentration in High Dimensions

24.1.1 Uniform distribution P on the unit ball $B_d = B(0, 1)$.

The volume of a ball centered at x with radius r:

$$Vol(B(x, r)) = Vol(B(0, r)) = r^d v_d\,,$$

where

$$v_d = Vol(B(0, 1)) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \sim (\frac{2\pi e}{d})^{d/2}\,.$$

P concentrates near the Boundary of B_d:

$$P(1 - \epsilon \leq ||X|| \leq 1) \geq 1 - e^{1\epsilon d}\,.$$

The intuition is that if any coordinate of x is close to the boundary then x is close to the boundary.

P concentrates around a "SLAB":

Define

$$SLAB = \{x \in B_d : |x_1| \leq \frac{c}{\sqrt{d}}\}.$$

Note that

$$P(X \in SLAB) = P(|X_1| \leq \frac{c}{\sqrt{d}}) = P(|u^T X| \leq \frac{c}{\sqrt{d}}) = P(||uu^T X|| \leq \frac{c}{\sqrt{d}}) = \frac{Vol(SLAB)}{v_d}$$

for all $u \in S^{d-1}$ where $S^{d-1} = \{x \in R^d : ||x|| = 1\}$.

Now consider $Vol(SLAB)$. For each $r \in (0, 1)$ the intersection between the hyperplane $\{x : x_1 = r\}$ and B_d is a $d - 1$ dimensional ball $B_{d-1}(r)$ with radius $\sqrt{1 - r^2}$ centered at $(r, 0, ..., 0)$. So

$$\begin{aligned} vol(SLAB) &= \int_{-c/\sqrt{d}}^{c/\sqrt{d}} Vol(B_{d-1}(r))dr \\ &= v_{d-1} \int_{-c/\sqrt{d}}^{c/\sqrt{d}} (1 - r^2)^{(d-1)/2} dr \end{aligned}$$

$$= v_{d-1} \int_{-c}^{c} (1 - t^2/d)^{(d-1)/2} \frac{1}{\sqrt{d}} dt \,.$$

Hence

$$P(X \in SLAB) = \frac{v_{d-1}}{v_d} \frac{1}{\sqrt{d}} \int_{-c}^{c} (1 - t^2/d)^{(d-1)/2} dt \,.$$

$v_d \sim (\frac{2\pi e}{d})^{d/2}$ gives

$$\frac{v_{d-1}}{v_d} \sim \frac{1}{\sqrt{2\pi e}} (\frac{d}{d-1})^{d/2} \sqrt{d-1} \,.$$

Since $(\frac{d}{d-1})^{d/2} \to e^{1/2}$ and $(1 - t^2/d)^{(d-1)/2} \to e^{-t^2/2}$ by dominating convergence theorem we have

$$P(X \in SLAB) \to \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} e^{-t^2/2} dt = P(|Z| \leq c) \,,$$

where $Z \sim N(0,1)$.

24.1.2 Uniform distribution P on the unit cube $[0,1]^d$.

Let H be the hyperplane orthogonal to a principal diagonal at the center of the cube:

$$H = \{x \in [0,1]^d \,:\, x^T e = \sqrt{d}/2\} \,, e = (\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}, ..., \frac{1}{\sqrt{d}}) \,.$$

Then X **concentrates around** H:

Let

$$H_{c\sqrt{d}} = \{x \in [0,1]^d \,:\, |x^T e - \sqrt{d}/2| \leq c\sqrt{d}\} \,.$$

Note that $x^T e = \sum_i x_i/\sqrt{d}$. So

$$P(X \in H_{c\sqrt{d}}) = P(x \in [0,1]^d \,:\, -c\sqrt{d} \leq \sum_i x_i/\sqrt{d} - \sqrt{d}/2 \leq c\sqrt{d}) \to 1$$

by to Central Limit Theorem

$$\frac{\sum_i x_i}{\sqrt{d}} - \frac{\sqrt{d}}{2} \to N(0, \frac{1}{12}) \,.$$

24.1.3 Uniform distribution δ_{d-1} on the unit sphere S^{d-1} in R^d.

$$S^{d-1} = \{x \in R^d \,:\, ||x|| = 1\} \,.$$

δ_{d-1} **concentrates around any equator** $E = \{x \in S^{d-1}, x_1 = 0\}$**:**

Let $E^\epsilon = \{x \in S^{d-1} \,:\, |x_1| \leq \epsilon\}$. We want to show

$$\delta_{d-1}(E^\epsilon) \geq 1 - 2e^{-d\epsilon^2/2} \,.$$

Let C^ϵ be the complement of E^ϵ in the upper hemisphere. Then

$$\delta_{d-1}(C^\epsilon) = \frac{Vol(cone(C^\epsilon))}{v_d} \,.$$

$cone(C^\epsilon)$ is covered in the ball $B(x', \sqrt{1-\epsilon^2})$ where $x' = (\epsilon, 0, ..., 0)$. Hence

$$Vol(cone(C^\epsilon)) \leq Vol(B(x', \sqrt{1-\epsilon^2})) = (1-\epsilon^2)^{d/2} v_d \leq v_d e^{-d\epsilon^2/2}\,,$$

which gives $\delta_{d-1}(C^\epsilon) \leq e^{-d\epsilon^2/2}$. Therefore,

$$\delta_{d-1}(E^\epsilon) = 1 - 2\delta_{d-1}(C^\epsilon) \geq 1 - 2e^{-d\epsilon^2/2}\,.$$