36-788, Fall 2015
Homework 1

Due Oct 1

1. (Among lipschitz functions the sum has the largest variance). Consider the class $\mathcal{F}$ of functions $f\colon \mathbb{R}^n \to \mathbb{R}$ that are Lipschitz with respect to the ℓ_1 distance, that is, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ then $|f(x) - f(y)| \leq \sum_{i=1}^n |x_i - y_i|$.Let $X = (X_1 \ldots, X_n)$ be a vector of independent random variables with finite variance. Use the EfronStein inequality to show that the maximal value of $\mathbb{V}(f(X))$ over $f \in \mathcal{F}$ is attained at the function $\sum_{i=1}^n X_i$. See Bobkov, S. and Houdre, C. (1996). Variance of Lipschitz functions and an isoperimetric problem for a class of product measures. Bernoulli, 2, 249255.

2. (Variance of supremuma of Guassian processes). Let T be a finite index set and let $(X_t)_{t\in T}$ be a centered Gaussian vector. Let $Z = \max_{t \in T} X_t$. Show that $\mathbb{V}(Z) \leq \max_{t\in T} \mathbb{V}(X_t)$.

3. Let Z be the number of triangles in a Erdös random graph $\mathcal{G}(n,p)$ (that is: each of the $\binom{n}{2}$ edges occur independently with probability p; also a triangle is a complete subgraph with 3 vertices). Calculate the variance of Z and compare it with the result obtained using the EfronStein inequality.

4. Let A be a symmetric matrix whose entries $X_{i,j}$ are independent random variables bounded in absolute value by 1, almost surely. Let

$$Z = \sup_{z\colon \|x\|=1} x^\top A x$$

denote its largest eigenvalue. Show that

$$\mathbb{V}\left[Z\right] \leq 16.$$

Proceed in this way. For all i and j let $A'_{i,j}$ the matrix obtained by replacing $X_{i,j}$ in A by the independent copy $X'_{i,j}$, while keeping all teh other variables fixed. Let $Z'_{i,j}$ denote the largest eigenvalue of $Z'_{i,j}$. Then

$$\left(Z - Z'_{i,j}\right)_+ \leq \left(v^\top (A - A'_{i,j}) v\right) 1_{Z > Z'_{i,j}},$$

where v is an eigenvector of Z, i.e. such that $Z = v^\top A v$. Continue bounding the above expression and the use Efron-Stein. See Example 3.14 in BLM.

5. (Convex poicare inequality). Let $X = (X_1, \ldots, X_n)$ be independent random variables taking values in $[0,1]$ and let $f\colon [0,1]^n \to \mathbb{R}$ be separately convex with existing partial derivatives. Then

$$\mathbb{V}[f(X)] \leq \mathbb{E}\left[\|\nabla f(X)\|^2\right].$$

Proceed in this way. Let $Z = f(X)$ and let $Z'_i = \inf_{x'_i} f(X_1, \ldots, X_{i-1}, x'_i, X_{i+1}, \ldots, X_n)$. In particular, let X'_i the value of x'_i for which the minimum is achieved (X'_i is a function of $\{X_j, j \neq i\}$). Then $Z_i = f(X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n)$. Now bound $(Z - Z_i)^2$ using separate convexity. See Theorem 3.17 in BLM.