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	SDS 387
	Linear Models
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	Lecture 5 - Tue, Sep 10, 2024 Contraction of the second
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	Instructor: Prof Ale Binaldo
	Convergence in distribution (weakest form of stochastic
	convergence)
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· · · · · · A	5 sequence of voindon vectors in R* 2×13 converges
· · · · · A	t sequence of voundary vectors in R* EX13 converges
ΑΑΑΑΑΑΑΑ	t sequence of voundary vectors in R* EX13 converges in dustribution (or weakly) to a roundary vector X
· · · · · · · · · · ·	t sequence of voundarin vectors in R* EX13 converges in dustribution (or weakly) to a roundarin vector X when the Rd st. Fx is contravas at 2
· · · · · · · · · · · ·	t sequence of voundari vectors in R* EX13 converges in dustribution (or weakly) to a roundari vector X when, tree Rd s.t. Fx is continuas at 2,
· · · · · · · · · · · · ·	t sequence of volumbon vectors in \mathbb{R}^{*} $\{X_{1}\}$ converges in distribution (or weakly) to a roundom vector X when, the \mathbb{R}^{d} s.t. F_{X} is continuas at \mathcal{Z} , \mathcal{A}^{l} of X
· · · · · · · · · · · · · · · · · · ·	t sequence of voundari vectors in R* EX13 converges in dustribution (or weakly) to a roundari vector X when, the Rd s.t. Fx is continuas at 2, all of X
	t sequence of voundari vectors in R* 2×13 converges in dustribution (or weakly) to a roundari vector X when, tree Rd s.t. Fx is continuas at a, ed/ of X
$X_{n} \xrightarrow{d} X_{n}$	t sequence of volumbon vectors in $\mathbb{R}^{\#} \ge \chi_{1}$ converges in distribution (or weakly) to a roundom vector χ_{1} when , the \mathbb{R}^{d} s.t. Fix is continual at z , all of χ_{1} some $F_{\chi_{n}}(z) \longrightarrow F_{\chi}(z)$ or $n \gg \infty$.
$X_{n} \xrightarrow{d} X$	t sequence of volumbon vectors in $\mathbb{R}^{\#} \ge \chi_{1}$ converges in distribution (or weakly) to a roundom vector χ_{1} when , the \mathbb{R}^{d} s.t. Fix is continued at z , all of χ_{1} some $F_{\chi_{n}}(z_{1}) \longrightarrow F_{\chi_{n}}(z_{2})$ or $n \rightarrow \infty$.
$X_{n} \xrightarrow{d} X$ $X_{n} \xrightarrow{>} X$	t sequence of voundari vectors in $\mathbb{R}^{\#} \ge \chi_{1}$ converges in distribution (or weakly) to a roundari vector χ when , $\forall x \in \mathbb{R}^{d}$ s.t. F_{χ} is continual at χ , $d\ell' = d\ell' = \chi$ $\int \mathcal{F}_{\chi_{n}}(x) \longrightarrow F_{\chi}(x)$ as $n \ge \infty$.
$X_n \xrightarrow{d} X$ $X_n \xrightarrow{q} X$ $X_n \xrightarrow{q} X$	t sequence of voundari vectors in $\mathbb{R}^{\#} \ge X_{1}$ converges in distribution (or weakly) to a roundari vector X_{1} when , $\forall x \in \mathbb{R}^{d}$ s.t. F_{X} is continual at x , $d = \int_{-\infty}^{U} f_{X}(x) \longrightarrow F_{X}(x)$ or $n \to \infty$. cold of X_{1}
$X_n \xrightarrow{d} X$ $X_n \xrightarrow{>} X$ $X_n \xrightarrow{>} X$	t sequence of volumbon vectors in \mathbb{R}^{*} $\{X_{1}\}$ converges in distribution (or weakly) to a roundom vector X when , the \mathbb{R}^{d} s.t. Fix is continued at a, all of X some $T = F_{X_{n}}(x) \rightarrow F_{X}(x)$ or $n \rightarrow \infty$. colf of X_{n} $\int pointwise convergence of \{F_{X_{n}}\} at all continuety$
$X_{n} \xrightarrow{d} X$ $X_{m} \xrightarrow{>} X$ $X_{m} \xrightarrow{\sim} X$	t sequence of volumbon vectors in \mathbb{R}^{*} $\{X_{1}\}$ converges in distribution (or vector) to a roundom vector X when , the \mathbb{R}^{d} s.t. Fx is continued at z , all of X some $T = F_{X_{n}}(z) \rightarrow F_{X_{n}}(z)$ or $n \Rightarrow \infty$. cut of X_{n} [pointwise convergence of $\{F_{X_{n}}\}$ at all continuety
$X_{n} \xrightarrow{d} X$ $X_{n} \xrightarrow{\rightarrow} X$ $X_{n} \xrightarrow{\rightarrow} X$	t sequence of volumbon vectors in $\mathbb{R}^{\#} \{ \{ X_{1} \} \}$ converges in distribution (or vector) to a roundom vector X when , the \mathbb{R}^{d} s.t. F_{X} is continued at z , all of X some $F_{X_{n}}(z) \rightarrow F_{X}(z)$ as $n \Rightarrow \infty$. cut of X_{n} [pointwise convergence of $\{ \{ F_{X_{n}} \} \}$ at all continuity point of F_{X}]
$X_n \xrightarrow{d} X$ $X_n \xrightarrow{X} X$ $X_n \xrightarrow{X} X$	t sequence of voindom vectors in \mathbb{R}^* { Xn} converges in distribution (or vector) to a roundom vector X when, the \mathbb{R}^d s.t. Fx is continues at 2, all of X some \mathbb{R}^* Fxn (2) \rightarrow Fx (2) as $n \rightarrow \infty$. colf of Xn [pointwise convergence of {Fxn} at all continuety point of Fx]
$X_{n} \xrightarrow{d} X$ $X_{m} \xrightarrow{>} X$ $X_{m} \xrightarrow{>} X$	t sequence of roundom vectors in R* EXiz converges in distribution (or vector) to a roundom vector X when, tree Rd s.t. Fx is continues at 2, when the Rd s.t. Fx is at all continues at 2, pointwise convergence of {Fx is at all continues at 2, point of Fx]
$X_n \xrightarrow{d} X \xrightarrow{r}$ $X_n \xrightarrow{r} X \xrightarrow{r}$ $X_n m \xrightarrow{r} X$ $Remore$	t sequence of voundari vectors in R* 2×13 converges in distribution (or weakly) to a roundari vector X when, the Rd s.t. Fx is continued at 2, d' of X some Fxin (2) ~ Fx (2) as n > co. colf of Xin [pointwise convergence of {Fxis} at all continuety point of Fx] This definition does not impose any restrictor
$X_n \xrightarrow{d} X$ $X_n X$ $X_n X$ $X_n X$ $X_n X$	t sequence of voindon vectors in R* EX13 converges in distribution (or veckly) to a roundon vector X when, tree Rd s.t. Fx is continued at 2, all of X Some FXin (2) -> FX(2) as n > co. colf of Xn [pointwise convergence of {Fxin} at all continuity point of Fx]
$X_n \xrightarrow{d} X$ $X_n \xrightarrow{>} X$ $X_n X$ $X_n X$ $Remore$	t sequence of voundari vectors in R* EX13 converges in distribution (or veakly) to a roundari vector X when, tree Rd s.t. Fx is continuated at 2, all of X Some Fxn (2) -> Fx (2) as n > co. colf of Xin [pointwise convergence of {Fxn3 at all continuity point of Fx] 2 ~) This definition does not impose any restriction on the joint distribution of the Xi's
$X_n \xrightarrow{d} X$ $X_n X$ $X_n X$ $X_n X$ $Remore$	t sequence of voundari vectors in \mathbb{R}^{*} [Xi] converges in distribution (or veckly) to a roundari vector X when, the \mathbb{R}^{d} s.t. Fix is continued at 2, all of X some $T = F_{Xin}(x) \longrightarrow F_{X}(x)$ as $n \Rightarrow \infty$. call of Xin [pointwise convergence of {F_{Xin}} at all continuety point of Fix] a) This definition does not impose any restriction on the jaint distribution of the Xin's
$\begin{array}{c} X_n \xrightarrow{d} X \\ \times n \xrightarrow{d} X \end{array}$ $\begin{array}{c} X_n \xrightarrow{d} X \\ \times n \end{array}$ $\begin{array}{c} X_n \xrightarrow{d} X \\ \times n \end{array}$ $\begin{array}{c} Remove$	t sequence of voundari vectors in \mathbb{R}^* [Xi] converges in distribution (or veckly) to a roundari vector X when, the \mathbb{R}^d s.t. Fx is continued at 2, all of X some $=$ Fxin (x) \rightarrow Fx (x) or $n \rightarrow \infty$. cut of Xin [pointwise convergence of {Fxin} at all continuity point of Fx] \rightarrow This definition does not impose any restriction on the jaint distribution of the Xi's aut of X
$X_n \xrightarrow{d} X$ $X_n X$ $X_n X$ $X_n X$ $X_n X$ $Remove$	t sequence of voundari vectors in \mathbb{R}^* $\{X_n\}$ converges in distribution (or vector) to a roundari vector X when , the \mathbb{R}^d s.t. Fx is continued at x , dt' d' X Some $T = F_{Xn}(x) \rightarrow F_{X}(x)$ as $n \rightarrow \infty$. cut of X_n $[$ pointwise convergence of $\{F_{Xn}\}$ at all continuety point of F_X] 2 a) This definition does not impose any restriction on the jaint distribution of the $Xn's$ and of X $[$ For avaluable $Xn = (-1)^n = 2n N(0it)$
$X_n \xrightarrow{d} X$ $X_n \xrightarrow{>} X$ $X_n \xrightarrow{>} X$ $X_n X$ $Remove$	t sequence of voindom vectors in R* EX13 converges in distribution (or veckly) to a roundom vector X when, the Rd s.t. Fx is continued at 2, all of X some Fxn (2) -> Fx (2) as n > co. cdf of Xn [pointwise convergence of {Fxn} at all continuity point of Fx]
$X_n \xrightarrow{d} X$ $X_n \xrightarrow{d} X$ $X_n \xrightarrow{d} X$ $X_n \xrightarrow{d} X$ $Remove$	t sequence of voundar vectors in \mathbb{R}^* {Xi} converges in distribution (or vector) to a roundar vector X when, the \mathbb{R}^d s.t. Fx is continued at 2, all of X $=$ Fxin (2) \rightarrow Fx (2) or $n \rightarrow \infty$. cut of Xin [pointwise convergence of {Fxin} ait all continuity point of Fx] = n) This definition does not impose any restriction on the joint distribution of the Xin's aux of X = $=$ $=$ $=$ $=$ $=$ $=$ $=$ $=$ $=$

· ·	and $X \sim N(O_{11})$ then $X_{1} \xrightarrow{d} X$ but $X_{1} \xrightarrow{p} X$ nor $X_{1} \xrightarrow{wp1} X$
	The fact that Xn and X have similar crip's abes not mean they take on similar values. In above
	essande an NNO.1) and announce) but in general [Xi - X] can be large
$\frac{\text{Examples}}{\text{Fn}(z)} = \begin{cases} \\ \\ \\ \end{cases}$	et $\overline{\Phi}$ be the citil of N(Diri) and lef $0 \pi < -n$ $\overline{\Phi(\pi)} - \overline{\Phi(-n)}$ $-n \leq \pi < n$ $\overline{\Phi(n)} - \overline{\Phi(-n)}$ $\pi \geq n$
Then Fo	$n \Rightarrow \phi$ pointaire
(x) $(x) =$	$\begin{cases} 0 & 1 < -\eta \\ 0 & -\eta & -\eta \\$
	ρS $v \sigma V$ $f + S c v e t c$. $1 \wedge - > \psi$

The restriction that converge tokes place at all
continuity points of Fx only is necessary (
Example $X_n = \begin{cases} i - \frac{i}{n} & wp & \sqrt{2} \\ 0 - \frac{i}{n} & wp & \sqrt{2} \end{cases}$ in order
$\begin{cases} l+1 & wp & l/2 \\ 0 \neq (& wp & l/2 \end{cases}$
It is clear that, for large u_1 $X_1 \approx \text{Bernoullic(12)}$
That is in fact the case. But for $n = 1$ $F_{X_n}(1) = \begin{cases} 1 & n & add \end{cases}$
Ls $F_{x_n}(i)$ aloes not converge !!
That is an : 21=1 is a point of discontinuity of the coff Bernoulli (122)
Result: If $X_n \xrightarrow{P} X$ then $X_n \xrightarrow{d} X$ $Pf / X_n \xrightarrow{P} X$ means that, $\forall \varepsilon > 0$
$P(Xn - X \perp \varepsilon) \rightarrow 0 op n \rightarrow \infty$ So, for only $z \in R$, $P(Xn - X \perp \varepsilon) \rightarrow 0 op n \rightarrow \infty$ $\subseteq \{X_n \leq z\}$ $P(X \leq z - \varepsilon\} \cap \{ Xn - X \leq \varepsilon\}) \perp$
$P(A) = P(ABB) + IP(ABB^{c}) \qquad \qquad$

$\leq IP(X_n \leq x) \neq IP(L \times n - \times L > \varepsilon)$
$\mathbb{P}(X \leq x - \varepsilon) - \mathbb{P}(X_n - X > \varepsilon) \leq \mathbb{P}(X_n \leq x)$
Similarly,
$(\mathfrak{K} \times) \qquad \mathbb{P}\left(X_{n} \leq n\right) \leq \mathbb{P}\left(X_{\leq n+\epsilon}\right) + \mathbb{P}\left(I \times (X_{n} - \times I) \right)$
Because $\lim \inf \mathbb{P}(X_n - X > s) = \lim \mathbb{P}(X_n - X > s) = 0$ inequalities (X) and (K M) imply
$(\mathbb{P}(X \leq x-x) \leq \lim \mathbb{P}(X_{in} \leq x) \leq \lim \mathbb{P}(X_{n} \leq x)$
$\mathcal{L} = \mathcal{L} = $
Let E VO to get:
$\mathbb{P}(X < x) \leq \lim_{n \to \infty} \lim_{n \to \infty} \mathbb{P}(X_n \leq x) \leq \lim_{n \to \infty} \mathbb{P}(X_n \leq x) \leq \mathbb{P}(X \leq x)$
$F_{x}(x^{-})$
If a is a continuity point of Fx, Fx(2-) = Fx(2)
s_{r} , $lin = F_{r}(z) = F_{r}(z)$, $n \rightarrow \infty$

Schematic of stachastic convergence.	•
$\begin{bmatrix} \chi_n \stackrel{\mu \rho 1}{\rightarrow} \chi \end{bmatrix} \qquad \begin{bmatrix} \chi_n \stackrel{L \rho}{\rightarrow} \chi \end{bmatrix}$	•
$X_{n} \xrightarrow{\rho} X$	•
$\begin{array}{c} & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & & \\ & & & \\ & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\$	•
Result: If Xn d X where X is degenerate (non-roundon)	•
then $\chi_n \xrightarrow{r} \chi$	•
$Pt/$ Let coll $X \approx [1.e. P(X=x)=1]$ They	•
we wont to show that $I^{(1)}(1 \times n - \times 1 \times 2) \longrightarrow 2$ $5mell$ Fix $5 \ge 0$ Theo	
$P(X_n - z \ge \varepsilon) \le P(X_n \le z - \varepsilon) + P(X_n \ge z + \varepsilon)$	
i = i + i + i + i + i + i + i + i + i +	
$\leq P\left(X_{n} \leq x - \varepsilon\right) + 1 - P\left(X_{n} \leq x + \varepsilon\right)$)
	•
	•
	•
(5)	•

<u>Remarks</u> :	$f \begin{bmatrix} x_n \\ y_n \end{bmatrix} \xrightarrow{d} \begin{bmatrix} x \\ y \end{bmatrix}$ then we can conclude
ioint weak to for the standard	not $X_n \stackrel{d}{=} X$ and $Y_n \stackrel{d}{=} Y$ the
weak convergence thouse of manginals but not the sther gener	ver, if Xin > X and Yn > Y, in rol we count courclude that
voy crovwi	$\begin{bmatrix} x_n \\ y_n \end{bmatrix} \xrightarrow{d} \begin{bmatrix} x \\ y \end{bmatrix}$
Example	Let Un Umform (oil)
	$Y_n = \begin{cases} 0 & \text{out} & n \\ 0 & n & \text{odd} \end{cases}$
Clearly	$X_n \xrightarrow{d} U$ $Y_n \xrightarrow{d} U$
But .	[Yn] obes not converge at all!
	Hue
	κ
	Let For be the cott of Uniform [-n, m]
	Does Fu converge? No! The issue is that the mass is too

dimension
About v): when $d=1$ toke $A = (-\infty, z]$
then IP(XINGA) = Finla) and
$P(X \in \partial A) = P(X = A)$
which is = 0 when a is a continuity point of Fx
More generally: Xn is a discret uniform on Eo, 1,, n-1]
Then $\frac{X_n}{n} \xrightarrow{d} U$ where $U_N U_{net}form (orc)$
$A = [o_{i}] \cap (p_{n})$ then $P(X_{n} \in A) = f$
but $P(UeA) = D$
Q: explain why condition v) is not violated !
Hu .
Continuous nopping theorem:
Let f: "R" -> R and let X be a roundon vector
in Rd s.t. P(XEC)=1 where C is the set
of continuity point of f. Then
$X_{\Lambda} \xrightarrow{*} X \Longrightarrow f(X_{\Lambda}) \xrightarrow{*} f(X)$
where x means up 1 or in probability or
M distribution-
Note: if f is continuous this as trivially true