SDS 387
Linear Models

Fall 2025

Lecture 1 - Tue, Aug 26, 2025

Instructor: Prof. Ale Rinaldo

RECAP OF DETERMINISTIC CONVERGENCE

$\to$ d is fixed!!

- We will be working in Euclidean space $\mathbb{R}^d$, equipped with an inner product:

$$x = \begin{bmatrix} x(1) \\ \vdots \\ x(d) \end{bmatrix} \qquad y = \begin{bmatrix} y(1) \\ \vdots \\ y(d) \end{bmatrix}$$

$$\langle x, y \rangle = x^T y = y^T x = \sum_{i=1}^{d} x(i)\, y(i)$$

giving the Euclidean norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$ and a

notion of distance: $\|x - y\|$

- Much of what we are going to say is valid more general settings. For example, we could consider

L_p norms: $\quad \|x\|_p = \left(\sum_{i=1}^{d} |x(i)|^p \right)^{1/p}$

$p \geq 1$

$$\|x\|_\infty = \max_j |x(j)|$$

- We could consider general metric spaces:

$(\mathcal{X}, d)$

set $\quad$ distance function

$d: \mathcal{X} \times \mathcal{X} \to [0, \infty]$
s.t.
$d(x,x) = 0$
$d(x,y) = d(y,x)$
$d(x,y) \leq d(x,z) + d(z,y)$

$\|x-y\|$ is a distance!

- Let $\{x_n\}_{n=1,2,\ldots}$ be a sequence of points in $\mathbb{R}^d$ (or $\mathcal{X}$). Then $x_n \to x \in \mathcal{X}$ as $n \to \infty$ when

Deterministic convergence

$$\lim_{n \to \infty} d(x_n, x) = 0$$

Assume a normed space

- Notation: Let $\{r_n\}_{n=1,2,\ldots}$ be a sequence of positive numbers

big-oh $\quad x_n = O(r_n) \iff \exists C > 0$ s.t. $\dfrac{\|x_n\|}{r_n} \leq C$ all n

$x_n = O(1)$ means $\{x_n\}$ is bounded

little-oh $\quad x_n = o(r_n) \iff \forall \varepsilon > 0$ (arbitrarily small) $\exists N(\varepsilon)$ s.t.

$$\frac{\|x_n\|}{r_n} \leq \varepsilon \quad \forall n > N(\varepsilon)$$

$x_n = o(1)$ means $x_n \to 0$

big-omega $\quad x_n = \Omega(r_n) \iff \exists C > 0$ s.t. $\dfrac{\|x_n\|}{r_n} \geq C$ all n

$x_n = \Omega(1)$?

little-omega $\quad x_n = \omega(r_n) \iff \forall M > 0$ (arbitrarily large) $\exists N(M)$ s.t.

$$\frac{\|x_n\|}{r_n} \geq M \quad \forall n \geq N(M)$$

$x_n = \omega(1)$?

- $x_n = \Theta(r_n) \iff x_n = O(r_n)$, $x_n = \Omega(r_n)$

$x_n = \Theta(1)$?

STOCHASTIC CONVERGENCE

capital letters denote random variables

- Suppose $\{X_n\}_{n=1,2,\dots}$ is a sequence of random vectors.

- Notation: for an event A (collection of possible outcomes) $\mathbb{P}(A)$ is the probability that A occurs

Example: $Z \sim N(0,1)$ $\quad A = \{|Z| > 1.96\}$

Almost sure or almost everywhere convergence
AKA convergence with probability 1

Let $\{X_n\}$ be a sequence of r.v.'s and X another random variable (possibly degenerate)

Then

$$X_n \xrightarrow{a.s} X \quad \text{or} \quad X_n \xrightarrow{a.e} X \quad \text{or} \quad X_n \xrightarrow{wp\ 1} X$$

when

$$\mathbb{P}\left(\left\{\lim_{n\to\infty} d(X_n, X) = 0\right\}\right) = 1$$

↓

the probability that a realization of the entire sequence and of X leads to deterministic convergence is 1 !

- This is a very strong form of convergence !
- Equivalently say :

$$\mathbb{P}\left(\limsup d(X_n, X) > \varepsilon \right) = 0$$

$\forall \varepsilon > 0$

$\hookrightarrow$ arbitrarily small

$\downarrow$

That is the probability that

$d(X_n, X) > \varepsilon$ infinitely often is zero.

- lim inf and limsup of events

Let, for $\varepsilon > 0$ ($\to$ small), $A_{n,\varepsilon} = \{ d(X_n, X) < \varepsilon \}$

Then $X_n \xrightarrow{wp\ 1} X$ iff $\forall \varepsilon > 0$

$$\mathbb{P}\Big(\underbrace{\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_{m,\varepsilon}}_{\liminf A_{n,\varepsilon}} \Big) = 1$$

or

$$\mathbb{P}\Big(\underbrace{\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A^c_{m,\varepsilon}}_{\limsup A^c_{n,\varepsilon}} \Big) = 0$$

$\liminf A_{n,\varepsilon}$: eventually $d(X_n, X) < \varepsilon$

$\exists N$ s.t. $A_{n,\varepsilon}$ is true for all $n \geq N$

$\limsup A^c_{n,\varepsilon}$: $d(X_n, X) \geq \varepsilon$ infinitely often

$\forall N$ $\exists N' \geq N$ s.t. $A^c_{N',\varepsilon}$ occurs