SDS 387
Linear Models

Fall 2025

Lecture 2 - Thu, Sept 4, 2025

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- Last time: convergence w.p. 1 (with probability 1)
  or
  a.e. (almost everywhere) convergence
  a.s. (almost surely)

- Recall $\{X_n\}_{n=1,2,\dots}$ sequence of r.v.'s

  X another r.v.

  Then $X_n \xrightarrow{\text{wp1, a.e., a.s.}} X$ as $n \to \infty$ when

$$\mathbb{P}\Big(\underbrace{\big\{ \lim_{n\to\infty} d(X_n, X) = 0 \big\}}_{\text{deterministic convergence}}\Big) = 1$$

Equivalently, $\forall \varepsilon > 0$ arbitrarily small

$$\mathbb{P}\Big(\big\{ \limsup_{n\to\infty} d(X_n, X) > \varepsilon \big\}\Big) = 0$$

↓

the probability that

$d(X_n, X) > \varepsilon$ infinitely often (or i.o.)

is zero

- liminf and limsup of a sequence of events

Let $\{A_n\}_{n=1,2,\dots}$ be a sequence of events for example take $A_n = \{d(X_n, X) < \varepsilon\}$ or $A_{n,\varepsilon}$

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \iff A_n \text{ happens i.o.}$$

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m \iff A_n \text{ happens eventually}$$

(i.e. A_n happens for <u>all</u> n large enough)

- For convergence wp 1: let, for a given small $\varepsilon > 0$

$A_{n,\varepsilon} = \{d(X_n, X) < \varepsilon\}$, then

$X_n \xrightarrow{wp\ 1} X$ means

$$\mathbb{P}(\limsup A_{n,\varepsilon}^c) = 0$$

$\left(\bigcup_n A_n\right)^c = \bigcap_n A_n^c$

$\left(\bigcap A_n\right)^c = \bigcup A_n^c$

$\Updownarrow$ De Morgan's Law

$$\mathbb{P}(\liminf A_{n,\varepsilon}) = 1$$

- Remark: Convergence wp 1 requires you to have some control / knowledge of the joint distribution of $\{X_n\}_{n=1,2,\dots}$ and X.

- A weaker and more useful notion of stochastic convergence is that of

Convergence in probability

$\{X_n\}_{n=1,2,\dots}$ and X → often degenerate (i.e. non stochastic)

small

$$\boxed{\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}\left(\{d(X_n, X) > \varepsilon\}\right) = 0}$$

written $X_n \xrightarrow{P} X$

we only need to control the distribution of X_n and X for each n.

Thm Convergence wp 1 implies convergence in probability.

Pf/ Let $C = \{\lim_{n\to\infty} d(X_n, X) = 0\}$. Then $X_n \xrightarrow{wp1} X$ is equivalent to $\mathbb{P}(C) = 1$

Let $\varepsilon > 0$ (small) and let

$$C_n = C_n(\varepsilon) = \{ d(X_k, X) \le \varepsilon, \ \forall k \ge n \}.$$

Then $C \subseteq \bigcup_{n=1}^{\infty} C_n$. So, $\mathbb{P}\left(\bigcup_{n=1}^{\infty} C_n\right) = 1$

Next, $C_n \subseteq C_{n+1}$, all n. Therefore

$\mathbb{P}(C_n) \to 1$ as $n \to \infty$

as $n \to \infty$.

Therefore

$$\lim_{n\to\infty} \mathbb{P}(C_n^c) = 0$$

But

$$\mathbb{P}\left(d(X_n, X) > \varepsilon\right) \le \mathbb{P}(C_n^c) \to 0 \text{ as } n \to \infty$$

<u>Fact</u>: CONTINUITY OF PROBABILITY

If $B_n \downarrow B = \bigcap_n B_n$

Then $\lim_n P(B_n) = P(B)$

If $B_n \uparrow B = \bigcup_n B_n$

Then $\lim_n P(B_n) = P(B)$

Because

$$\mathbb{P}(\liminf B_n) \le \liminf P(B_n) \le \limsup P(B_n) \le \mathbb{P}(\limsup B_n)$$

<u>Example</u> (the typewriter sequence)

Let $U \sim \text{Uniform}(0,1)$. Define $\{X_n\}$ as follows. For any $n \in \mathbb{N}_+$ we have that

$$2^k \le n < 2^{k+1}, \quad k = k(n) = \lfloor \log_2 n \rfloor$$

and define

$$X_n = f_n(U) = \begin{cases} 1 & \text{if } U \in \left[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}\right] \\ 0 & \end{cases}$$

So $X_1 = 1$ $X_2 = 1$ iff $U \in [0, 1/2]$

$X_3 = 1$ iff $U \in [1/2, 1]$ $X_4 = 1$ iff $U \in [0, 1/4]$

$X_5 = 1$ iff $U \in [1/4, 1/2]$ $X_6 = 1$ iff $U \in [1/2, 3/4]$

Q: $X_n \xrightarrow{P} 0$? Yes !!

For any $\varepsilon \in (0,1)$

$$\underbrace{\mathbb{P}(|X_n| > \varepsilon)}_{d(X_n, X) > \varepsilon,\ X = 0} = \mathbb{P}\left(U \in \left[\frac{n-2^k}{2^k}, \frac{n-2^k-1}{2^k}\right]\right)$$

$$= \frac{1}{2^k} \to 0$$

because $k \to \infty$ as $n \to \infty$

Q: $X_n \xrightarrow{wp\,1} 0$? No !!

Because for any $u \in (0,1)$

$f_n(u) > \varepsilon$ i.o.

<u>Example</u> Let $\{U_n\} \overset{iid}{\sim}$ Uniform $(0,1)$ and let

$$X_n = \begin{cases} 1 & \text{if } U_n \in [0, 1/n] \\ 0 & \text{otherwise} \end{cases}$$

$$= \mathbb{1}\{U_n \in [0, 1/n]\}$$

Q: $X_n \xrightarrow{P} 0$?

Yes because $\forall \varepsilon \in (0,1)$

$$\mathbb{P}(|X_n| \geq \varepsilon) = \mathbb{P}(U_n \in [0, 1/n]) = \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$

Q: $X_n \xrightarrow{wp\ 1} 0$? No !

Let's see why this is the case (non trivial !).
We will show that $\forall \varepsilon \in (0,1)$ $\mathbb{P}(\{|X_n| < \varepsilon \text{ eventually}\}) = 0$

$$\mathbb{P}(\{|X_n| < \varepsilon \text{ eventually}\}) = \mathbb{P}(\liminf_n \underbrace{A_{n,\varepsilon}}_{\{|X_n| < \varepsilon\}})$$

$$= \mathbb{P}\left(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_{m,\varepsilon}\right)$$

$$\leq \sum_{n=1}^{\infty} \mathbb{P}\left(\bigcap_{m=n}^{\infty} A_{m,\varepsilon}\right)$$

Union bound or countable sub-additivity: if $\{B_n\}$ is a sequence of events then $\mathbb{P}(\bigcup_n B_n) \leq \sum_n \mathbb{P}(B_n)$

Next $\mathbb{P}\left(\bigcap_{m=n}^{\infty} A_{m,\varepsilon}\right) = \lim_{k \to \infty} \mathbb{P}\left(\bigcap_{m=n}^{k} A_{m,\varepsilon}\right)$

by continuity of probability

Let's now study $\mathbb{P}\left(\bigcap_{m=n}^{k} \underbrace{A_{m,\varepsilon}}_{\{X_m < \varepsilon\}}\right)$

$$= \mathbb{P}(\{|X_m| < \varepsilon \text{ all } n \leq m \leq k\})$$

$$= \prod_{m=n}^{k} \left(1 - \frac{1}{m}\right)$$ by indepence of the U_n's

$\hookrightarrow$

$$\mathbb{P}\left(\bigcap_{m=n}^{\infty} A_{m,\varepsilon}\right) = \lim_{k\to\infty} \prod_{m=n}^{k} \left(1 - \frac{1}{m}\right)$$

fact $1-x \leq e^{-x}$

$$\leq \lim_{k\to\infty} \exp\left\{-\sum_{m=n}^{k} \frac{1}{m}\right\}$$

$$= 0$$

because $\sum_{m=1}^{\infty} \frac{1}{m} = \lim_{k\to\infty} \sum_{m=1}^{k} \frac{1}{m} \sim \lim_{k\to\infty} \log k = \infty$

So

$$\mathbb{P}\left(\{|X_n| < \varepsilon \text{ eventually}\}\right) \leq \lim_{k\to\infty} \sum_{n=1}^{k} \underbrace{\mathbb{P}\left(\bigcap_{m=n}^{\infty} A_{m,\varepsilon}\right)}_{0}$$

$$= 0$$

∎

We have just proved:

Borel - Cantelli's Second Lemma If $\{A_n\}$ is a sequence of independent events and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ then $\mathbb{P}\left(\limsup_n A_n\right) = 1$

<u>Example</u> $\{X_n\}$ are independent with

$X_n \sim \text{Bernoulli}(p_n)$

$p_n \in (0,1)$

$$\mathbb{P}(\{X_n = 1 \text{ i.o.}\}) = ?$$

If $\sum p_n = \infty$ it is 1 !