SDS 387
Linear Models

Fall 2025

Lecture 7 - Thu, Sept 23, 2025

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- Are the stochastic limits (i.e. wp 1, $\xrightarrow{P}$ and $\xrightarrow{d}$) unique?

Yes!

- For $\xrightarrow{d}$, let $\{X_n\}$ be a r.v.'s s.t.

$$X_n \xrightarrow{d} Y \quad \text{and} \quad X_n \xrightarrow{d} Z$$

Then $Y \stackrel{d}{=} Z$, that is $F_Y = F_Z$. To see this let D_Y and D_Z be the points of discontinuities of F_Y and of F_Z, respectively. We know that D_Y and D_Z are countable sets, and is $D_Y \cup D_Z$.

So $\forall x \in \mathbb{R} \setminus (D_Y \cup D_Z)$

set minus $[A \setminus B = A \cap B^c]$

$$F_{X_n}(x) \longrightarrow F_Y(x) = F_Z(x) = F(x)$$

$\hookrightarrow$ F_Y and F_Z coincide at $\mathbb{R} \setminus (D_Y \cup D_Z)$

So now take $x \in D_Y \cup D_Z$.

Then $\exists \{x_n\}$ s.t. $x_n \notin D_Y \cup D_Z \;\forall n$

s.t. $x_n \downarrow x$

By right continuity of F_Y and F_Z

$$\lim_n F_Y(x_n) = F_Y(x), \quad \lim_n F_Z(x_n) = F_Z(x)$$

But $F_Y(x_n) = F_Z(x_n)$ for all n because $x_n \notin D_Y \cup D_Z$

$\downarrow$

$\lim_n F(x_n)$ exists $= F_Y(x) = F_Z(x) = F(x)$.

- Total variation between 2 probability distributions, say P and Q in $\mathbb{R}^d$ is

$$TV(P,Q) = \sup_{A \subseteq \mathbb{R}^d} |P(A) - Q(A)|$$

Borel sets

$$= \frac{1}{2} \int_{\mathbb{R}^d} |p(x) - q(x)|\, dx$$

if p and q are the pdf of P and Q

- convergence in TV is stronger that $\xrightarrow{d}$

$X_n \xrightarrow{d} X \Rightarrow F_{X_n}(x) \to F_X(x)$ for all continuity point of X

$P_{X_n}(A) \to P_X(A)$ for all sets

$A = (-\infty, x]$ x continuity point of F_X

and $P_{X_n}(A) = \mathbb{P}(X_n \in A)$

$X_n \xrightarrow{TV} X$ when $P_{X_n}(A) \to P_X(A)$ for <u>all</u> (Borel) sets A!

Portmanteau Theorem

(See Van der Vaart's book chapter 2)

- Let's call $\xrightarrow{d}$ weak convergence

↳ see, eg., Weak convergence of prob. measure by P. Billingsley

- The following are equivalent (and equivalent to $X_n \xrightarrow{d} X$)

i) $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$ for all bounded continuous real valued functions

ii) $\liminf_n \mathbb{P}(X_n \in G) \geq \mathbb{P}(X \in G)$

for all open sets G

iii) $\limsup_n \mathbb{P}(X_n \in G) \leq \mathbb{P}(X \in G)$

all closed sets G

iv) $\mathbb{P}(X_n \in A) \to \mathbb{P}(X \in A)$ all Borel sets

A s.t. $\mathbb{P}(X \in \partial A) = 0$

↳ boundary

where $\partial A = \bar{A} \setminus A^\circ$

closure of A: smallest closed set containing A (intersection of all sets containing A)

interior of A: set of all points $x \in A$ s.t. $\exists B(x, \varepsilon)$ contained in A. open set union of all open balls in A.

About iv): if $A = (-\infty, x]$ and F_X is continuous at x then $\mathbb{P}(X \in \partial A) = 0$

↳ this gives us the condition

$F_{X_n}(x) \to F_X(x)$ for all continuity points x of F_X

<u>Example</u>: Let X_n be uniform on $\{k/n,\ k=0, \dots, n-1\}$

Then $F_{X_n}(x) = \dfrac{\lfloor nx \rfloor + 1}{n} \to F_X(x) = x$

$\forall x \in [0,1]$

F_X cdf Uniform$(0,1)$

Let $A = [0,1] \cap \mathbb{Q}$ → rationals

$\mathbb{P}(X_n \in A) = 1$ all n

but

$\mathbb{P}(X \in A) = 0$

$\partial A = [0,1]$

As an application, let's look at

- <u>Continuous mapping theorem</u>. Let $f: \mathbb{R}^d \to \mathbb{R}$ and

let C be the set of continuity points of f.

Let $\{X_n\}$ be a sequence of r.v.'s s.t.

$X_n \xrightarrow{d,\,p,\,wp1} X$

Then if $\mathbb{P}(X \in C) = 1$

true if f is continuous

$f(X_n) \xrightarrow{d,\,p,\,wp1} f(X)$

Pf/ only of $\xrightarrow{d}$ part. Let G be a closed set in $\mathbb{R}$ and notice that $\{f(X_n) \in G\} =$

$$\{X_n \in \underbrace{f^{-1}(G)}_{\{x : f(x) \in G\}}\}$$

Next:

$$(*) \quad f^{-1}(G) \subseteq \overline{f^{-1}(G)} \subseteq f^{-1}(G) \cup C^c$$

To see the second inclusion, take $x \in \overline{f^{-1}(G)}$

Then $\exists \{x_n\} \subset f^{-1}(G)$ s.t. $x_n \to x$ and $f(x_n) \in G$ all n. If $x \in C$ then

$f(x_n) \to f(x) \in G$ because G is closed $[\Rightarrow x \in f^{-1}(G)]$

otherwise $x \in C^c$.

So

$$\limsup_n \mathbb{P}(f(X_n) \in G) = \limsup_n \mathbb{P}(X_n \in f^{-1}(G))$$

$$\leq \limsup_n \mathbb{P}(X_n \in \overline{f^{-1}(G)})$$

by portmanteau part (iii)

$$\leq \mathbb{P}(X \in \overline{f^{-1}(G)})$$

by $(*)$

$$\leq \mathbb{P}(X \in f^{-1}(G)) + \underbrace{\mathbb{P}(X \in C^c)}_{\substack{=0 \\ \text{by assumption}}}$$

$$= \mathbb{P}(X \in f^{-1}(G))$$

$$= \mathbb{P}(f(X) \in G)$$

$\hookrightarrow$ we have shown that for every closed set G in $\mathbb{R}$

$$\limsup_n \mathbb{P}(f(X_n) \in G) \leq \mathbb{P}(f(X) \in G)$$

By Portmanteau part (iii) again

$$f(X_n) \xrightarrow{d} f(X)$$

∎

<u>Remark</u>: The continuity condition $\mathbb{P}(X \in C) = 1$ is important

Suppose $X_n \xrightarrow{d} 0$ and let

$$f(x) = \mathbb{1}\{x \geq 0\}$$

Does it mean $f(X_n) \xrightarrow{P} 1$?

Not necessarily

$$X_n = \begin{cases} 0 + \frac{1}{n} & 1/2 \\ 0 - \frac{1}{n} & 1/2 \end{cases}$$

Then $X_n \xrightarrow{P} 0$ but $f(X_n) \not\xrightarrow{P} 1$