SDS 387
Linear Models

Fall 2025

Lecture 14 - Thu, Oct 16, 2025

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- Positive (semi)-definite matrices: $\underset{d \times d}{A}$ symmetric is p.s.d when $\underbrace{x^T A x}_{} \geq 0$ $\forall x \in \mathbb{R}^d$

it is positive definite if $x^T A x > 0$ $\forall x \in \mathbb{R}^d$

$$\sum_{i,j} A_{i,j} x_i x_j = \sum_i A_{i,i} x_i^2 + 2 \sum_{i<j} A_{ij} x_i x_j$$

(Remark: if A is not symmetric use $A + A^T$)

- A has spectral decomposition $A = U \Lambda U^T$

U orthogonal, Λ diagonal containing eigenvalues

↳ A is psd iff $\lambda_i \geq 0$ all i

(λ_i: eigenvalues)

A is nsd (negative semi definite) iff $\lambda_i \leq 0$ all i

- Positive semi-definite order: partial order over square symmetric matrices. $A \succsim B$

whenever $A - B \succeq 0$. Because this is a partial order $A - B \not\succeq 0$ does not imply that $A - B \prec 0$

■ SVD singular value decomposition

Let $\underset{m \times n}{A}$ be a general matrix. Notice that $\underset{n \times n}{A^T A}$ and $\underbrace{AA^T}_{m \times m}$ are psd and have the same eigenvalues

The positive square root of those eigenvalues are the singular value of

The SVD of $\underset{m \times n}{A}$ wrt $\text{rank}(A) = r \le \underbrace{\min\{m, n\}}_{:= q}$

is given by

$$\underset{m\times n}{A} = \underset{m \times m}{U} \underset{m \times n}{\Sigma_1} \underset{n \times n}{V}^T$$

where $\underset{m \times m}{U}$ $\underset{n \times n}{V}$ orthogonal

and

$$\underset{m \times n}{\Sigma_1} = \begin{cases} \underset{q \times q}{\Sigma_q} & \text{if } m = n = q \\ \begin{bmatrix} \Sigma_q & \underset{n-m}{0} \end{bmatrix} & q = m \\ \begin{array}{c} \\ m-n \end{array} \begin{bmatrix} \Sigma_q \\ 0 \end{bmatrix} & q = n \end{cases}$$

and

$$\underset{q \times q}{\Sigma_q} = \text{diag}(\sigma_1, \ldots, \sigma_q)$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \underbrace{\sigma_{r+1} = \cdots = \sigma_q}_{=0} = 0$

are the singular values

- A simpler form may be:

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

u_i: ith column of U
v_i: ith column of V

The u_i's are the left singular vectors

v_i's are the right singular vectors

- $\boxed{\text{The first } \min(m,n)}$ columns of U span $R(A)$

" V " $R(A^T)$

- <u>Remarks</u> : if $\underset{n \times n}{A}$ psd then singular value are the eigenvalues

and (left or right) singular vectors are eigenvectors

also

largest eigenv. $\leftarrow$ $\lambda_{max}(A) = \sigma_1 = \max_{x:\|x\|=1} x^T A x$

$$\lambda_{min}(A) = \sigma_n = \min_{x : \|x\| = 1} x^T A x$$

- if $\underset{n \times n}{A}$ square but not symmetric then

$$\sigma_1 = \max_{x : \|x\| = 1} |x^T A x|$$

- In general if $\underset{m \times n}{A}$ then

$$\sigma_1(A) = \max_{\substack{x \in \mathbb{R}^m \\ \|x\| = 1}} \max_{\substack{y \in \mathbb{R}^n \\ \|y\| = 1}} x^T A y$$

- if $\underset{\substack{n \times n \\ \text{symmetric}}}{A}$ then: $A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k \text{ times}}$ is

$$= \underset{n \times n}{U} \underset{n \times n}{\Lambda} \underset{n \times n}{U^T} \qquad = U \Lambda^k U^T$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad \Lambda^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i u_i u_i^T$$

if, say, $\lambda_1, \ldots, \lambda_r$ are $\neq 0$ and $\lambda_{r+1}, \ldots, \lambda_n = 0$

$$= U' \Lambda' U'^T$$

$$\underset{n \times r}{U'} \qquad \Lambda' = \text{diag}(\lambda_1 \ldots \lambda_r)$$

- Let $f: \mathbb{R} \to \mathbb{R}$, $\underset{\substack{n\times n \\ \text{symmetric}}}{A} = U \Lambda U^T$ then

$$f(A) = U \begin{bmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{bmatrix} U^T$$

- If $A \succeq 0$ $\exists Q \succeq 0$ s.t. $A = Q^2$

$$Q = A^{1/2} \quad \text{square root of } A$$

PROJECTION AND PROJECTION MATRICES

Let N be a linear subspace in $\mathbb{R}^d$.
The for any $x \in \mathbb{R}^d$ the orthogonal projection of x onto N is the unique point $y \in N$ s.t.

$$y = \underset{z \in N}{\arg\min} \; \| x - z \|$$

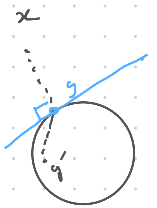
Also $x - y \in N^\perp$

This is because $x = \underbrace{x_N}_{y} + x_{N^\perp}$.

- We are using the standard inner product $\langle x, y \rangle$ to define orthogonality

→ unique

- The projection is well defined over a larger class of convex sets. If C is a closed convex set in $\mathbb{R}^d$ and $x \notin C$ then the projection, say y, of x onto C is unique and satisfy



$$\langle x - y, y' - y \rangle \le 0 \quad \forall y' \in C$$

angle btw x, y θ

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \, \|y\|}$$

- Projections can be found using projection matrices or projectors. If N is a linear subspace in $\mathbb{R}^d$ of dimension $r \le d$ then the $\overline{\text{orthogonal}}$ projection of x onto N is given by

$$y = \underset{d \times d}{\underset{\sim}{P}} x \quad \text{where } P \text{ satisfies}$$

i) $P^2 = P$ idempotent (projection property)

ii) P is symmetric

In fact properties i) and ii) defines projection matrices

- Let $\{v_1, \dots, v_r\}$ be an orthonormal basis for N. Then

$$P = VV^T = V I_d V^T \qquad \underset{d\times r}{V} = [v_1 \; . \; . \; . \; v_r]$$

$$= \sum_{i=1}^{r} v_i v_i^T \quad \text{so that}$$

$$Px = \sum_{i=1}^{r} \langle v_i, x\rangle \, v_i$$

$$\left(\|Px\|^2 = \sum_{i=1}^{r} \langle v_i, x\rangle^2 \right)$$

- If $\underset{d\times r}{V}$ s.t. $R(V) = N$ then

$$P = V (V^T V)^{-1} V^T \qquad \text{(symmetric and idempotent)}$$

- P has eigenvalues that are 0 or 1 !!

- Property i) $[P^2 = P]$ is a projection property:

we can define a projection on N as a mapping P s.t. when restricted to N

$$P \circ P = P \quad \text{i.e.}$$

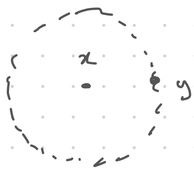
$$P(P(x)) = P(x)$$

- If we use a different inner product:

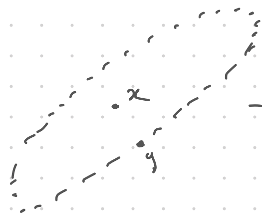
$$\langle x, y\rangle_\Sigma = x^T \Sigma y \qquad \underset{\text{pd}}{\Sigma} \text{ matrix}$$

this defines a new norm $\|x\|_{\Sigma^{-1}} = \sqrt{x^T \Sigma^{-1} x}$

$\|x - y\|_{\Sigma^{-1}}$ Mahalanobis distance



Euclidean distance



→ Mahalanobis distance